Unit 4
Trigonometry 2
(15-20%)

Trigonometry 2 -

Outcomes

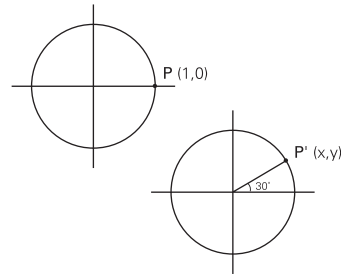
SCO: By the end of Mathematics 2204/2205, students will be expected to

C9 analyse tables and graphs of various sine and cosine functions to find patterns, identify characteristics, and determine equations

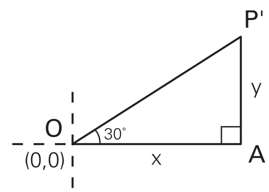
A1 demonstrate an understanding of irrational numbers in applications

Elaboration – Instructional Strategies/Suggestions

C9/A1 Students will develop an understanding of the relationships among the angle of rotation, the radius of the circle, and the coordinates of a point that is on the circumference of the circle. Students should begin by watching the point P (1, 0) as it rotates around the unit circle centred at (0, 0). Students should rotate the point P, 30° and determine its coordinates. To determine the coordinates, they would drop a perpendicular from P' to the x-axis and find the length of the two legs in the right triangle. The x-coordinate for would be the horizontal distance from the centre (0, 0) to the foot of the perpendicular A.

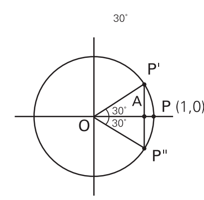


Since the radius , then *x* can be determined: . Similarly, . Students should conclude that when by rotating 30° about centre (0, 0) on a unit circle, the coordinates for will be (cos 30°, sin 30°). These coordinates can be expressed as decimal values: and $\sin 30° \doteq 0.5$, or students might determine these values using transformations and the Pythagorean Theorem: Reflect across the x-axis and join and . The is an equilateral triangle and . Since each side measures 1 unit then $AP' = \frac{1}{2}$ unit and



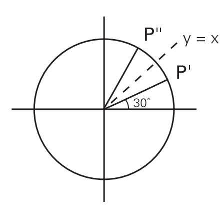
$(OA)^2 = (OP')^2 - (P'A)^2$ (Pythagorean Theorem).

$$OA = \sqrt{1-\left(\frac{1}{2}\right)^2}$$
$$= \pm\sqrt{1-\frac{1}{4}}$$
$$= \pm\frac{\sqrt{3}}{2}$$



So, the coordinates for after a 30° rotation about (0, 0) in a unit circle can be expressed as (cos 30°, sin 30°) or approximately as (0.866, 0.5), or

exactly as $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$, using the positive value .



Reflecting across *y* = *x* produces the Point . This location of can be expressed as a rotation of 60°, about centre (0, 0) of the point (1, 0) on the unit circle. Since P'' now, is the image of after a reflection in *y* = *x*, then its coordinates are the reverse of the coordinates of or (0.5, 0.866) or exactly as $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$.

continued ...

Trigonometry 2

Worthwhile Tasks for Instruction and/or Assessment

C9/A1

Pencil/Paper

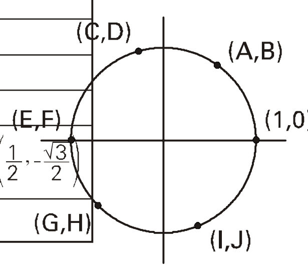
1. Complete the following table for a rotation of P(1, 0), about centre (0, 0):

Angle of rotation	Image of P (1,0)		
	As (x,y)	As a decimal	Exact
30°			$\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$
60°	(cos60°, sin60°)		
		(-0.5, 0.866°)	
	(cos270°, sin270°)		
120°			
			$\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$
180°			

2. i) Locate the following points on the unit circle:
 a) (cos 330°, sin 330°) b) (cos -120°, sin -120°)
 c) (cos 180°, sin 180°) d) (cos 210°, sin 210°)

 ii) Write the co-ordinates for (i) as exact values.

3. Which coordinate could represent the value of each of the following:



a) cos(290°) b) sin(180°) c) cos(110°)
d) sin(225°) e) cos(–70°) f) cos(420°)

A1

Journal

4. Explain why $\frac{1}{2}$ can be expressed as an exact value, but $\frac{\sqrt{3}}{2}$ is only approximately 0.866.

Suggested Resources

Trigonometry 2 - Irrational Numbers

Outcomes

SCO: By the end of Mathematics 2204/2205, students will be expected to

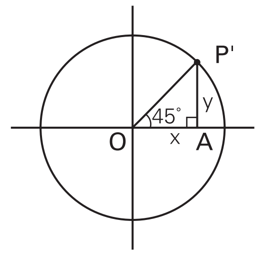
C9 analyse tables and graphs of various sine and cosine functions to find patterns, identify characteristics, and determine equations

A1 demonstrate an understanding of irrational numbers in applications

Elaboration – Instructional Strategies/Suggestions

C9/A1 A 45° rotation of P(1, 0) about centre (0, 0) on the unit circle would result in P' being right on the line y = x forming an isosceles when the perpendicular is dropped from .

but since P' is on $y = x$, then $y = x$ and



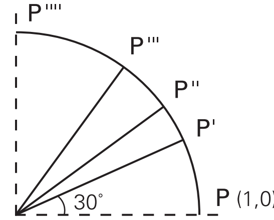
but $OP' = 1$, so

Students should be able to simplify: $\sqrt{\frac{1}{2}} = \frac{\sqrt{1}}{\sqrt{2}} = \frac{1}{\sqrt{2}}$ and by convention the denominator be rationalized: . In this form students are more able to think of this coordinate as 'half of root 2' or about 0.7. So, the coordinates for P' after a 45° rotation about (0, 0) on the unit circle is or

approximately (0.707, 0.707) or exactly , again using the positive value

for .

To quickly summarize, students should now be able to relate some of the rotations of P (1, 0), and the coordinates of P through those rotations on the unit circle (radius 1). Students should develop strategies that will allow them to interplay between the relationships. Mentally they should be able to determine the values for cos 45°, sin 60°, cos 0°, sin 90°, etc. They should be able to say that if the coordinate for R is (0.707, 0.707), R is located at the point that is the image of P(1, 0) after a 45° rotation, about centre (0, 0) on the unit circle.



Trigonometry 2

Worthwhile Tasks for Instruction and/or Assessment

Suggested Resources

C9/A1

Pencil/Paper

1. Complete the following table for a rotation of P(1, 0), about centre (0, 0):

angle of rotation	Image of P (1,0)		
	As (x,y)	As a decimal	Exact
45˚			
	(cos225˚, sin225˚)		
		(.707, -.707˚)	
	(cos270˚, sin270˚)		
			$\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$
330˚			
			$\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

2. i) Locate the following points on the unit circle.
   a) (cos 45°, sin 45°) b) (cos 240°, sin 240°)
   c) (cos 315°, sin 315°) d) (-0.707, 0.707)
   ii) Write the above coordinates as exact values.

$\frac{\sqrt{2}}{2}$ $\frac{1}{2}$ $\sin(135°) = \frac{\sqrt{2}}{2}$

3. Explain why .

4. Explain why $\cos(-60°) = \frac{1}{2}$ and not .

A1

Journal

5. Explain why is called an exact value, while 0.707 is only approximate.

Trigonometry 2 -

Outcomes

SCO: By the end of Mathematics 2204/2205, students will be expected to

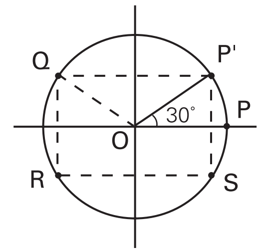
C9 analyse tables and graphs of various sine and cosine functions to find patterns, identify characteristics, and determine equations

A1 demonstrate an understanding of irrational numbers in applications

C15 demonstrate an understanding of sine and cosine ratios and functions for non-acute angles

Elaboration – Instructional Strategies/Suggestions

C9/A1/C15 Students should extend this understanding of the relationships just established by visualizing the locations of these points as they are reflected across the x- and y-axes to locations in the 2nd, 3rd, and 4th quadrants. In these new locations the same values exist for the multiples of 30°, 60°, and 45° angle rotations though the signs are dependent of the quadrant. Note how Q is the image of reflected in the y-axis. The or 150°. So, the coordinates for Q are the same for P', but the x-value is



negative (–0.866, 0.5) or . Similarly R

becomes and S . Using their knowledge of these relationships students can evaluate trigonometric expressions like

to get and simplify to $\frac{3\sqrt{2}}{2}-\frac{1}{4}\Rightarrow\frac{6\sqrt{2}-1}{4}$.

(Notice $\sin^2 30°$ in the above notation. Students should understand that this is the common way of writing $(\sin 30°)^2$.

Trigonometry 2

Worthwhile Tasks for Instruction and/or Assessment

C9/A1/C15

Pencil/Paper

6. The point P(1, 0) on a unit circle centred at the origin is rotated through an angle of –120° about (0, 0).
   a) Find the coordinates of the image point of P and write a mapping rule for this rotation.
   b) Evaluate cos (-120°) and sin (-120°).

7. Name the smallest positive angle of rotation which is the same as each of the following rotations.
   a) -540° b) -270°
   c) -390° d) -135°

8. The image of P(1, 0) on the unit circle is rotated about (0, 0). State the angle of rotation for each of the following image coordinates.

   a) b) c) (-1, 0) d)

9. Evaluate the following:
   a) sin 30° + cos 60° b) 12 sin 45° cos 45°
   c) 20 sin 60° cos 240° d) cos 180°cos 45° – sin 180° sin 45°

   e) $\sin^2 30° + \cos^2 70°$ f)

   g) $\frac{\cos 45°}{\sin 120°} - \frac{\sin 210°}{\cos 30°}$

C9/A1

Journal

10. If and , explain why cos must be .

Suggested Resources

Trigonometry 2 - Trigonometric Equations

Outcomes

SCO: By the end of Mathematics 2204/2205, students will be expected to

C1 model situations with sinusoidal functions

C28 analyse and solve trigonometric equations with and without technology

C18 interpolate and extrapolate to solve problems

C30 demonstrate an understanding of the relationship between solving algebraic and trigonometric equations

B4 use the calculator correctly and efficiently

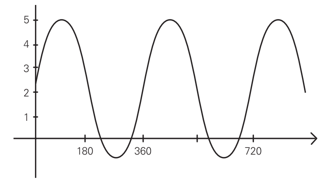
C27 apply function notation to trigonometric equations

Elaboration – Instructional Strategies/Suggestions

C1/C28/C18 All students should understand that the relationship being studied by graphing trigonometric functions is the relationship between the coordinates for the points on the unit circle and the angle rotation. If students graph y = sin θ and trace, they will see, for example, when $\theta = 30$, that y = 0.5. Students should now be able to use the equation of the sinusoidal relationship to predict or calculate values to help solve problems.

C30 As students model situations with sinusoidal equations they are expected to obtain approximate answers by interpolation and extrapolation, or by tracing the graph. They should extend this knowledge to getting more exact answers by solving these equations both with and without technology.

For example, the graph of the rotation of a water wheel is given. Students could find the equation and use the equation to answer a question like "how high is a certain paddle after 2 full rotations?"



The equation: . After 2 rotations $(360° \times 2)$ they would evaluate sin 720°. Its value is the same as sin 360° or , that is, the value is zero. Consider the unit circle. If P (1, 0) does not rotate, or rotates one complete or 2 complete rotations the sine is still zero. Students would then solve and conclude that the paddle is 2 m high after 2 full rotations.

$$\frac{1}{3}(y-2)=0$$

$$y-2=0 \text{ (multiplying both sides by 3)}$$

$$y=2 \text{ (adding 2 to both sides)}$$

C30/B4 If students were asked how many degrees rotation would result in the paddle having a height of –0.5 m (underwater), then students would solve

$$\frac{1}{3}(y-2)=\sin\theta, \text{ and substitute -0.5}$$

$$\frac{1}{3}(-.5-2)\cdot\ \sin\theta \qquad (y=-0.5)$$

$$-\frac{2.5}{3}=\sin\theta$$

$$\theta=\sin^{-1}\left(-\frac{2.5}{3}\right) \text{(taking inverse sine of both sides)}$$

56°

This answer implies that if the wheel rotated –56° the paddle would be 0.5 m below the waterline. Students will learn on the next two-page spread that there are infinitely more correct answers, since this relationship is periodic.

C30/C27/B4 Sometimes trigonometric equations are given in the form . Students should recognize this as different organization of the terms, and that it could be expressed as a function . Students should be able to evaluate the function using function notation. For example, they should be able to evaluate f(45°), f(20°)+f(120°), etc. They should also be able to determine , given , etc.

Trigonometry 2

Worthwhile Tasks for Instruction and/or Assessment

C28/B4 *Activity*

1.The following activity will help students understand the connection between the unit circle and the graphs of and .

a) Have students create a unit circle and mark the points (1, 0), (0, 1), (-1, 0), and (0, -1). Label (1,0) as P.

b) Have students sketch the first period of beside the unit circle so that 1 on the vertical axis is the same scale as 1 on the unit circle. Students should label the horizontal axis in degrees.

c) Below the sketch of have students sketch so that the axes correspond (same scale).

d) Ask students to label the point P on their two graphs, and explain why it is labelled where it is.

e) Ask students to rotate P to (0, 1) and label this .

f) Ask students to locate on the two other graphs. Have students explain their thinking.

g) Ask students to rotate P to (0, -1) and label this , and locate this on their two graphs.

h) Ask students to use their graphs to find the coordinates for if is the image of P after a 180° rotation. Ask students to explain how they obtained the coordinates from the two graphs. (Check using the unit circle).

[illegible]

i) Ask students to use the graphs to determine the co-ordinates for the image of P as it rotates from (1, 0) through the following rotations.

i) 45° ii) 270° iii) 210° iv) 300°

C28/C27 *Performance*

2. Graph the functions below and show/explain how you would use the graph to evaluate

a) *f*(22.5°) b) *f*(90°) c)

d) e)

i) ii)

C28/C30/B4

3.Show how to evaluate *f*(22.5°) and *f*(–90°) for both equations above using an algebraic approach.

4.The equation represents the relationship between the height of a frog sitting on a paddle above the water as the water wheel rotates.

a) What is the diameter of the wheel?

b) What do "*h*" and " " stand for?

c) What would be a proper domain and range?

d) How high above the water does the frog get?

e) Describe the rotation of the wheel in as many ways as you can.

f) When does the frog first reach 10 units above the water?

g) How much must the wheel rotate for the frog to come out of the water?

Suggested Resources

Trigonometry 2 - Trigonometric Equations

Outcomes

SCO: By the end of Mathematics 2204/2205, students will be expected to

C1 model situations with sinusoidal functions

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C30 demonstrate an understanding of the relationship between solving algebraic and trigonometric equations

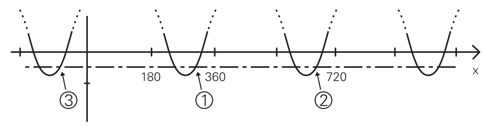
B4 use the calculator correctly and efficiently

Elaboration – Instructional Strategies/Suggestions

C1/C28/C18/C30/B4 On the previous two-page spread, students solved

for y = -0.5, and

obtained the answer . The graph doesn't show any negative degrees along the -axis. Students might have been expecting an answer around 200°. In fact, if students drew a horizontal line through y = -0.5, it would intersect the graph in several places. If 56° was subtracted from 360° the intersection point ① would be the point that answers the problem. This point repeats itself every +360° or –360° as can be seen on the graph, marked ② and ③. Students should then show the solution to the equation

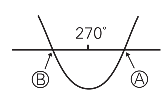


mentioned above as: $\theta = \sin^{-1}\left(-\frac{2.5}{3}\right)$

$$\theta = -56° + 360°k, k \in I$$

Also, students can see another rotation that results in a depth of –0.5 m. They need to remember the symmetry around this local minimum value. The x-value directly above the local minimum would be 270° (half-way between 180° and 360°). Since the x-value at A is ≐ 304°, that is 270° + 34°, so then B is ≐ 270° – 34° or 236°. Now students should complete

the solution above as: $\theta = \sin^{-1}\left(-\frac{2.5}{3}\right)$



$$\theta \doteq 270 \pm 34 + 360\,k,\ k \in \mathrm{I}$$
$$\theta \doteq \{236,\ 304\} + 360\,k,\ k \in \mathrm{I}$$

C30/B4 Where y-level students have the option to solve trigonometric equations using graphing techniques, or using technology, or algebraic methods, z-level students need to practice using algebraic techniques to build their manipulation skills. Z-level students thus will be expected to solve more complex trigonometric equations such as the following algebraically:

1.

$$2\sin 2\theta = -\sqrt{2}, 0 \le x \le 720°$$

$$\sin 2\theta = \frac{-\sqrt{2}}{2} \qquad k^{\circ}$$

$$2\theta = \begin{cases} 225° + 360' k^{\circ} \\ 315° + 360° \end{cases}$$

$$\theta = \begin{cases} 112.5° + 180°k, 0 \le k \le 4 \mid k \in I \\ 157.5° + 180°k, 0 \le k \le 4 \mid k \in I \end{cases}$$

2.

$$\sin^2\theta \qquad \sin\theta + 2 = 0, \text{ for all } \theta$$

$$(\sin\theta - 2)(\sin\theta - 1) = 0$$

$$\sin\theta = 2\varnothing \qquad \sin\theta = 1$$

$$\theta = 0 \qquad \theta = 90° + 360°k, k \in I$$

Trigonometry 2

Worthwhile Tasks for Instruction and/or Assessment

Suggested Resources

C28/C30/B4 *Pencil/Paper*

1. Show all your steps as you solve the following equations algebraically for

   .

   a) b)

   c) d)

2. (Adv-level) Solve the following for all θ:

   a) b)

   c) d) $\cos^2\theta = \sin\theta(1+\sin\theta)$

   e) $24 - 24\cos\theta = 7\sin\theta + 5$

3. Adv-level students should revisit #4 p. 115 on the previous third column and determine the answers to (d), (f) and (g) algebraically.

C1/C18

4. Tommy has a tree swing near the river in his backyard. The swing is a single rope hanging from a tree branch. When Tommy swings he goes back and forth across the shore of the river. One day his mommy (who was taking an adult math course) decided to model his motion using her stopwatch. She finds that after 2 seconds, Tommy is at one end of his swing, 4 metres from the shore, over land. After 6 seconds, he reaches the other end of his swing, 5.2 metres from the shore over the water.

   [illegible]

   a) Sketch a graph of this sinusoidal function.
   b) Write the equation expressing distance from the shore versus time.
   c) Predict the distance when
      i) time is 6.8 seconds
      ii) time is 15 seconds
      iii) time is 30 seconds
   d) Where was Tommy when his mommy started the watch?

C28/C30/B4 *Journal*

In order to solve the equations in #4(b) time must be expressed in terms of angle rotation. Assume the maximum angle from vertical formed by the rope of the swing is 120° and describe how to convert the time periods to accumulated angle measures. Write the equation expressing distance versus angle measure and find the angle measure from the vertical when Tommy is 1 metre from the shore, over land.

Trigonometry 2 - Identities

Outcomes

SCO: By the end of Mathematics 2204/2205, students will be expected to

C24 derive and apply the reciprocal and Pythagorean identities

Elaboration – Instructional Strategies/Suggestions

C24 Students should examine tables of values to determine relationships that exist among the trigonometric functions. Students should be asked to examine the following table and describe a relationship between and cosecant θ ($\csc\theta$), between and and between and .

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\csc\theta$	$\sec\theta$	$\cot\theta$
0	0		0	undefined	1	undefined
30°	$\frac{1}{2}$			2	$\frac{2\sqrt{3}}{3}$	$\sqrt{3}$
45°	$\frac{\sqrt{2}}{2}$		1	$\sqrt{2}$	$\sqrt{2}$	1
60°	$\frac{\sqrt{3}}{2}$		$\sqrt{3}$	$\frac{2\sqrt{3}}{3}$	2	$\frac{\sqrt{3}}{3}$

Students might notice a pattern when comparing sin 0° = 0 and csc 0° undefined, and

and csc 30° = 2. They might conjecture that the cosecant values for the

same are reciprocals of the sine values. They might then test this using

. The reciprocal is $\frac{2}{\sqrt{2}}$ and rationalized, is . The

trigonometric ratio for csc45° is . So the conjecture checks. Students should

know these as the reciprocal identities:

Students should complete the table for cosine θ values then determine the other relationships asked for. Students might also notice or be encouraged to find that

. They should explore more and agree that

Students should be asked to use the unit circle to show that $\sin^2\theta + \cos^2\theta = 1$ and call this a Pythagorean identity. They should then derive the other two Pythagorean identities: Divide

$$\frac{\sin^2\theta}{\sin^2\theta} + \frac{\cos^2\theta}{\cos^2\theta} = \frac{1}{\sin^2\theta}$$
$$\Rightarrow 1 + \cot^2\theta = \csc^2\theta$$

Trigonometry 2

Worthwhile Tasks for Instruction and/or Assessment

C24 *Pencil/Paper*

1. Given and ≐ ,
   a) state the product sec 32°(csc32°) as a decimal to four places. [ans: 2.2248
   b) evaluate

      i) [ans: 1.60033] ii) [ans: 0.624869]

   c) state the sum [ans: 2.2248 to four decimal places.

   d) Find a way to describe how this verifies that and $\cos\theta = \frac{1}{\sec\theta}$.

2. a) Draw a unit circle centred at (0, 0). Label the origin 0.
   b) Rotate P(1, 0) through $\theta°$, where . is the image point.
   c) Drop a perpendicular from to the x-axis. Call the x-intercept A.
   d) State the length of OA and as trigonometric expressions.
   e) Calculate the slope of .
   f) Describe how this verifies that .

$\frac{\sin A}{\csc A}+\frac{\cos A}{\sec A}=1$

3. Simplify the following expressions.

   a) $\cos\theta\tan\theta$ b)

   c) $\sin\theta+\cos\theta\tan\theta$ d)

   e) f) $\frac{1}{\sec^2\theta}+\frac{1}{\csc^2\theta}-1$

   g)

4. Verify the following identities.

   a) $\sec\theta-\tan\theta=\frac{1-\sin^2\theta}{\cos\theta}$ b) $\frac{1-\tan^2\theta}{1+\tan^2\theta}=1-\tan\theta$

   c) $\frac{\sec\beta}{\cos\beta}-\frac{\tan\beta}{\cot\beta}=1$ d)

Suggested Resources

Trigonometry 2 -

Outcomes

SCO: By the end of Mathematics 2204/2205, students will be expected to

C24 derive and apply the reciprocal and Pythagorean Identities

C25 prove trigonometric identities

Elaboration – Instructional Strategies/Suggestions

C24/C25 Students should understand that trigonometric identities (the relationships described on pg 116) are useful in the simplification of more complex trigonometric expressions. Students will apply identities to simplify expressions. Z-level students should have the opportunity to do more applications of identities and in more complex situations. Students will also be expected to examine more complex trigonometric statements and to prove or verify that the statements are true, or that they are also identities. For example, students might be asked to prove that this statement is true: .

Typically, students would begin with the more complicated side of the statement and simplify it to look like the other side. Strategies are used to help students think their way through each verification. For example, in looking at the left side (LHS): $\cos\theta - \sin^2\theta\cos\theta$, students should decide that it is factorable: $\cos\theta\left(1-\sin^2\theta\right)$, then the Pythagorean Identity can be used to replace . Since: they would then write $\cos\theta\left(\cos^2\theta\right)$. This combines through multiplication to give which is the same as the right side (RHS) of the original statement. The identity is verified. When written by the students, encourage them to be neat and to organize their verification. It might look something like this:

Verify if $\cos\theta - \sin^2\theta\cos\theta = \cos^3\theta$

LHS:	RHS:
$\cos\theta\left(1-\sin^2\theta\right)$ (factor)	$\cos^3\theta$
$\cos\theta\left(\cos^2\theta\right)$ (Pythagorean)	
$\cos^3\theta$ (same as RHS)	

Students should examine several strategies that will help them as they apply their identities in simplification or verification problems.

Here are some strategies:

1) Use reciprocal identities to express all trigonometric terms as sines and cosines.
2) Use Pythagorean identities.
3) Factor when possible.
4) Express two fractions as one.
5) Express one fraction as two.
6) Multiply by a clever form of one.
7) Multiply by conjugates.

Sometimes students will be helped by simplifying one side, then the other side until both sides are equal.

Trigonometry 2

Worthwhile Tasks for Instruction and/or Assessment

C24/C25

Performance

1. a) Use factoring to re-express $\sin^4\theta - \cos^4\theta$ in lowest terms.
   b) Use a Pythagorean identity to simplify further.

2. a) Express these two fractions as one fraction:
   b) Use a Pythagorean identity to simplify further.

3. a) Begin with $\dfrac{\sin\theta}{1+\cos\theta}$
   b) Multiply the numerator and denominator by $1-\cos\theta$.
   c) Why could this strategy be called multiplying by a different form of 1?
   d) Why could this strategy also be called multiplying by the conjugate?
   e) Simplify to make this fraction equal to .

4. a) Use the strategy of making one fraction two fractions with .

   [illegible] $= -\csc\phi\cot\phi$ $= 2\csc\alpha$ $\sin\theta\sin\alpha$

   b) Simplify and restate $\cot^2\theta$ as $\dfrac{\cos^2\theta}{\sin^2\theta}$, then simplify again.
   c) Continue until this simplifies to $-\cot^2\theta$. What other strategies did you use?

5. Prove that each equation is an identity. Describe the strategies that you use.
   a) $\tan\theta(\cot\theta+\tan\theta)=\sec^2\theta$
   b) $\cos^4\beta-\sin^4\beta=1-2\sin^2\beta$
   c) $\dfrac{1}{\sec^2\theta}+\dfrac{1}{\csc^2\theta}=1$
   d)
   e) $\sec^2\theta+\csc^2\theta=\sec^2\theta\csc^2\theta$
   f)

Suggested Resources

Trigonometry 2 -

Outcomes

SCO: By the end of Mathematics 2204/2205, students will be expected to

B1 demonstrate an understanding of the relationship between operations on fractions and rational algebraic expressions

Elaboration – Instructional Strategies/Suggestions

B1 Since many identities involve working with fractional form, this may be a good time to strengthen students' skills with combining, multiplying, dividing, and simplifying rational algebraic expressions. For example, ask students to combine

. Watch as they re-express with a common denominator: and then

simplify. Next give them a similar one with algebraic terms: . They should

follow the same steps: $\frac{9+10}{6x}$, then simplify. Give them two fractions to multiply

. Remove common factors before multiplying so that the numbers are

small: . Some might multiply the remaining factors $\frac{30}{40}$, then

factor and reduce again, leaving . Others could do all the reducing in the first step

and arrive at the same answer in one step.

Algebraically, when simplifying an expression like the following, students should factor any expressions where possible, then using division, simplify as much as possible:

B1 *Pencil/Paper*

Trigonometry 2

Worthwhile Tasks for Instruction and/or Assessment

1. Select some equations from below to show that you know how to combine algebraic fractions.

a) b) c)

d) $\frac{x^2}{4}+\frac{y^2}{5}$ e) $\frac{3m}{2a}+\frac{5m}{3b}$ f) $\frac{a^2+3}{2}+\frac{3a-a^2}{3}$

g) $\frac{m+2}{5}+\frac{m-2}{3}$ h) $\frac{4x-3}{15y}+\frac{12}{6y^2}$ i) $\frac{3m-5}{m^2}+\frac{4m}{m^3}$

j) $\frac{m^2-1}{2m}+\frac{m^2+1}{3m}$ k) $\frac{7x+9}{3y}+\frac{3x-2}{9y^2}$ l) $\frac{4}{8a}+\frac{5}{9a^2}$

m) $\frac{16-y}{4y}+\frac{3+y}{7y}$ n) $\frac{\sqrt{2}}{5}-\frac{3\sqrt{2}}{10}$ o) $\frac{\sqrt{18}}{2}-\frac{\sqrt{8}}{3}$

2. Select from below to show that you know how to multiply/divide algebraic fractions.

a) b)

c) $\frac{6a^2-a-1}{4a^2-3a-1}\times\frac{20a^2-a-1}{6a^2-5a+1}$ d)

e) f) $\frac{15x^2+13x+2}{8x^2+18x-35}\div\frac{18x^2-9x-14}{8x^2+22x-21}$

[illegible]

g) h)

i) j) $\frac{m}{m-7}\div\frac{m^3+6m^2+5m}{m-7}$

k) l) $\frac{m^4-m^2-6}{m^4+2m^2}\times\frac{m^2-7}{m^4+5m^2}$

3. a) In question 2(g) above, substitute the value zero for “*m*” in the expression and simplify. What do you notice?

   b) Try again, but use $m = -1$.

   c) $m = 0$ and $m = -1$ are called restrictions on the variable. Can you explain what that means?

   d) What restrictions are on “*m*” in question 2(l)? Explain.

4. Simplify. What values of the variables are not possible?

a) $\frac{1}{a^2+5a+6}+\frac{1}{a^2-9}$ b)

c) d)

e) f) $\frac{4}{3a^2+27a+60}+\frac{3}{2a^2+16a+30}$

Suggested Resources

Trigonometry 2 -

Outcomes

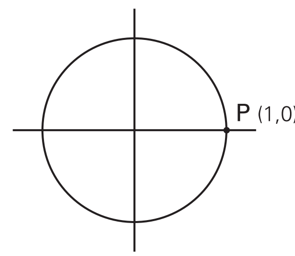
SCO: By the end of Mathematics 2204/2205, students will be expected to

D1 derive, analyse, and apply angle and arc length relationships

Elaboration – Instructional Strategies/Suggestions

D1 The final element in the development of trigonometric relationships is the arc-length a point travels as it is rotated through various angles of rotation, centre (0, 0) in the unit circle. All students should examine this element to determine the relationship between angle of rotation and arc length.

As the point P rotates clockwise (CW) around the circumference of the unit circle it travels certain distances. These distances are called arc lengths and can easily be calculated. For example, if P is rotated, centre (0, 0), 360°, it will travel the entire circumference. Students can determine the circumference, using . Since d = 2 in the unit circle, the entire circumference is 2 units. A 180° rotation would be



units, a 90° rotation units (written and read "pi over two"). Students should be given the opportunity to combine arc length into their other relationship understandings and complete a detailed table like the following:

angle of rotation	coordinates for image of P (1,0)	radius	arc length
0°	(1,0)		0
30°	$\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$		$\frac{30}{360}$ *or* $\frac{\pi}{6}$
150°	$\left(\frac{-\sqrt{3}}{2}, \frac{1}{2}\right)$		$\frac{5\pi}{6}$
60°		1	
300°		1	
	$\left(-\frac{\sqrt{2}}{2}, -\frac{2}{2}\right)$		
		1	$\frac{3\pi}{4}$

continued ...

Trigonometry 2

Worthwhile Tasks for Instruction and/or Assessment

D1

Pencil/Paper

1. Complete the following table for a rotation of P(1, 0) about (0, 0).

angle of rotation	coordinates for the image of P (1,0)	arc length
30°	$\left(\frac{\sqrt{3}}{2},\frac{1}{2}\right)$	$\frac{\pi}{6}$
45°	$\left(\frac{\sqrt{2}}{2},\quad\right)$	
		$\frac{\pi}{3}$
90°		
	$\left(\frac{\sqrt{3}}{2},\frac{1}{2}\right)$	
135°		
210°		
	$\left(\frac{\sqrt{2}}{2},-\frac{\sqrt{2}}{2}\right)$	
		$-\frac{2\pi}{3}$

Journal

2. Explain what these mean:

   a) An arc length of $-\frac{3\pi}{2}$ units is associated with an angle of rotation of –270°.

   b) An arc length of 6 units on the unit circle is related to some angle rotation.

Suggested Resources

Trigonometry 2 - Radians

Outcomes

SCO: By the end of Mathematics 2204/2205, students will be expected to

D1 derive, analyse, and apply angle and arc - length relationships

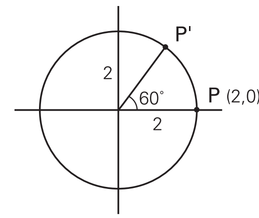
D2 demonstrate an understanding of the connection between degree and radian measure and apply them

C9 analyse tables and graphs of various sine and cosine functions to find patterns, identify characteristics and determine equations

Elaboration – Instructional Strategies/Suggestions

... continued

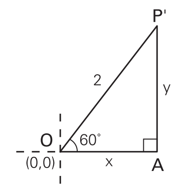
D1 Students should then examine how the trigonometric relationships are affected if the radius of the circle changes. They should remember from their study of dilatation that if the radius doubles, then so will the circumference. What happens to the coordinates of the image of P (1, 0)?



Begin with a circle radius 2 units and the point P (2, 0). Rotate it 60° about (0,0). Determine the coordinates for

the image of P which is (P'). Since in

$\Delta P'OA$ then x = 2cos 60°. Similarly, y = 2sin 60°. So, the coordinates for will be (2cos 60°, 2sin 60°). Thus, the trigonometric ratios are directly related to the radius. So if the radius is 3, the coordinates will be 3 times the value they would be on the unit circle, and the arc-length would be 3 times as long. Students should be able to explain why.



D2/C9 Up to this point in our study, when exploring the sine and cosine functions, the domain of the function has been expressed using degree measure.

Another unit of angle measure was developed long ago that comes from the relationships being studied between arc length and radius. This measure of rotation is called radians. This is the unit of choice in virtually all higher level mathematics and science.

Angle measure (in radians) = arc length radius

The angle measure or rotation is directly related to the # of radius lengths contained in an arc length. For example, an arc travel of 6 cm on a circle of radius 3 cm gives:

Notice there are no units on the 2. The 2 shows that the angle formed cm this circle would be exactly the same as 2 radians on any circle. This is why an angle measure written without units is understood to be in radians.

Trigonometry 2

Worthwhile Tasks for Instruction and/or Assessment

Suggested Resources

D1/D2 *Activity*

1. a) Using the origin as centre, draw three circles, radius 1, 2, 3 units.
   b) Make P (1, 0) on the unit circle.
   c) Rotate P (1, 0) 45°, centre (0, 0) and mark its image .
   d) State the coordinates for
   e) Draw a line from the origin O through to hit the circumferences of the other two circles at R and S.
   f) i) What is ? [ans: 45°]
      ii) What is ? [ans: 45°]
      iii) What is ? [ans: 45°]
   g) i) What is the arc length of ? Explain why.
      ii) Show how to determine the corresponding arc lengths on the other two circles.
   h) If has coordinates (cos 45°, sin 45°), what are the coordinates for R and S in terms of cosines and sines? Show how to determine these.
   i) From the relationship between the coordinates seen in part (h) state the exact coordinates for R and S in simplified form.
   j) Find a way to justify these coordinates using right triangle calculations.
   k) The angle of rotation was 45°. What is the corresponding rotation in radians?
   l) State the relationship that seems to be true about the angle measure, arc-length, radius, and corresponding coordinates on the circumferences of circles of different radii.

$\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

Performance

2. a) The point (1, 0) on the unit circle is rotated about the origin through each angle given in radian measure. For each rotation, find the arc length, to one decimal place, through which the point (1, 0) has moved.

   i) ii) iii) iv)

   v) vi) vii) viii)

   b) Redo (c), (d), and (g) above with P (2, 0) being rotated, centre (0, 0).

3. Find the length of the intercepted arc for each of the following central angles in a circle of radius 6cm. Express your answer to the nearest tenth of a centimetre.

   a) b) c) d)

4. Find the radian measure of the central angle that intercepts each of the following arcs in a circle of radius 5 cm.
   a) 12 cm b) 20 cm c) 5 cm d) 42 cm

5. The length of an arc of a circle intercepted by a central angle of 150° is 21 cm. Find the diameter of the circle correct to the nearest centimetre.

Trigonometry 2 - Radians

Outcomes

SCO: By the end of Mathematics 2204/2205, students will be expected to

D2 demonstrate an understanding of the connection between degree and radian measure and apply them

C9 analyse tables and graphs of various sine and cosine functions to find patterns, identify characteristics and determine equations

Elaboration – Instructional Strategies/Suggestions

...continued

D2/C9 Radians are numbers that belong to the Real number family. On the unit circle, the radian measure is the same as the arc length. Students should spend some time visualizing, for example, a 45º rotation in the unit circle as a rotation of radians. They might do this by completing a table like the following, while visualizing the unit circle or its image:

angle of rotation		coordinates for the image of P (1,0)		
degrees	radians		radius	arc length
30˚			2	
		$\left(\frac{-\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$	1	
210˚			2	
			1	$\frac{3\pi}{4}$
120˚			3	
. .	$\frac{5\pi}{6}$	. .	1	. .

All students should apply these relationships to problems involving angle measure and arc length.

Trigonometry 2

Worthwhile Tasks for Instruction and/or Assessment

D2/C9 *Pencil/Paper*

1. Complete the following table.

angle of rotation		coordinates for the image of P (1,0)	circle	
degrees	radians		radius	arc length
	$\frac{\pi}{6}$		3	
	$\frac{3\pi}{4}$		2	
210˚			1	
315˚				$\frac{7\pi}{4}$
	$\frac{2\pi}{3}$		3	
150˚			2	
	$\frac{7\pi}{4}$	$\left(\sqrt{2}, -\sqrt{2}\right)$		. .

2. Evaluate.

a) [illegible] b) [illegible] c) [illegible]

[illegible] $+\cos\frac{5\pi}{4}$ … $\sin\left(-\frac{\pi}{6}\right)$

d) [illegible] e) [illegible] f) $\sin^2\frac{\pi}{6}+\cos^2\frac{\pi}{6}$

g) $\left(\sin\frac{\pi}{6}\cos\frac{\pi}{3}\right)-\left(\cos\frac{\pi}{6}\sin\frac{\pi}{3}\right)$ h) $\cos\frac{2\pi}{2}-\sin\left(-\frac{\pi}{3}\right)$

i) [illegible]

Suggested Resources

Trigonometry 2 -

Outcomes

SCO: By the end of Mathematics 2204/2205, students will be expected to

D2 demonstrate an understanding of the connection between degree and radian measure and apply them

A1 demonstrate an understanding of irrational numbers in applications

B4 use the calculator correctly and efficiently

C10(Adv) analyse tables and graphs of various sine and cosine functions to find patterns, identify characteristics, and determine equations using radians

C4(Adv) determine the equations of sinusoidal functions expressed in radians

C22(Adv) describe how various changes in the parameters of sinusoidal equations, expressed in radians, affect their graphs

C16(Adv) demonstrate an understanding of sine and cosine ratios and functions for non-acute angles expressed in radians

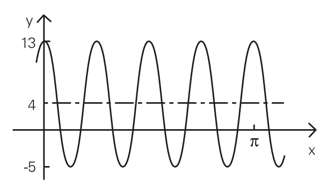
Elaboration – Instructional Strategies/Suggestions

C9/A1/B4 All students should be given the opportunity to evaluate trigonometric expressions given in degrees and radians. For example, all students can evaluate:

These kinds of questions provide all students with opportunities to strengthen their application of the relationships between radian measure, arc-length and coordinates of points, as well as build their mental math and visual skills. For example, for the question above, students should visualize the point P(1, 0) rotated $\frac{\pi}{4}$ on the unit circle, read its x-coordinate and double it . Then students would visualize P(1, 0) rotated and read its y-coordinate and triple it . Then students would have to demonstrate their ability with fractions and radicals to combine and , giving . The mental math and the exact answers should be encouraged. Advanced students are expected to become as proficient with radians as they are with degrees. The radian measure will become more important to students since radian values belong to the Real number system, and calculus work with trigonometric functions is simplified when the domain is the set of real numbers.

C10(Adv)/C4(Adv)/C22(Adv)/C16(Adv) Advanced students should revisit all types of questions and function work and re-explore and apply radian measure. For example, they should become very proficient with evaluating complex trigonometric expressions with radian measure.

They should be able to determine equations of sinusoidal situations with a domain in real numbers. To obtain the equation of this sinusoid, students would describe the transformations of $y = \cos x$, as a vertical translation of 4, vertical stretch of 9, no vertical shift, but a horizontal stretch of . (The period of



$y = \cos x$ is 2π, the period here is units, and .

Trigonometry 2

Worthwhile Tasks for Instruction and/or Assessment

Suggested Resources

D2/A1 *Performance*

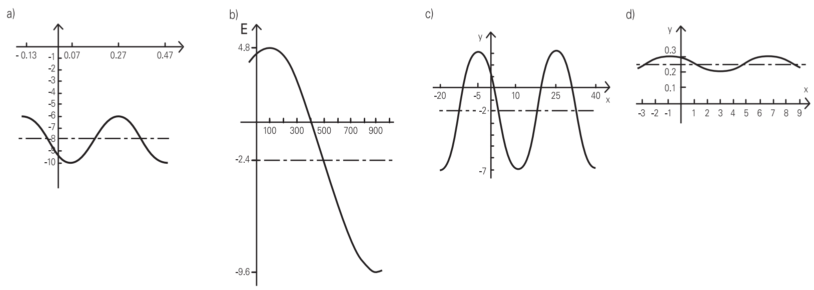
1. Using the unit circle to visualize, determine the numerical value for each expression.

   a)   b) $3\cos^2\frac{7\pi}{6}$ c)

   d)   e) $\dfrac{\sin\left(-\frac{\pi}{3}\right)}{\sin\left(\frac{3\pi}{4}\right)+2\cos\left(\frac{7\pi}{6}\right)}$

C10(Adv)/C4(Adv)

2. a) Determine the equations for the following graphs.



[illegible]

b) State the following for each of the above graphs: domain and range, zeros, equation for sinusoidal axis, values that cause decreasing portions of the graph, minimum values and where they occur.

C22(Adv)

3. a) Draw the graph given the following equations.

   i) $-(y+2)=\cos\left(x+\frac{3\pi}{4}\right)$ ii)

   b) State the domains and ranges, list the zeros, and give the values that cause maximum values for each of the above.

   c) State the mapping rule for each of the above if they are images of y = cos x and y = sin x respectively.

C16(Adv)

4. Evaluate:

   a)   b) $\sec\frac{4\pi}{3}-\tan\frac{7\pi}{6}$

   c) $5\sin^2\left(\frac{11\pi}{6}\right)-3\cos^2\left(\frac{5\pi}{3}\right)$ d)

C4(Adv) *Journal*

5. Revisit problem 4, p. 11 and explain why and how you could solve the equation in #4b to determine the amount of time that it takes for Tommy to be 1m from the shore, over land. What other times will this occur? Explain.

Trigonometry 2 -

Outcomes

SCO: By the end of Mathematics 2204/2205, students will be expected to

C29(Adv) analyse and solve trigonometric equations with and without technology, expressing solutions in radians

B4 use the calculator correctly and efficiently

Elaboration – Instructional Strategies/Suggestions

C29(Adv)/B4

When solving equations, advanced students should be able to solve equations in both degrees and radians. (See discussion on p. 112, 114 for solving equations in degrees.) Students need to be careful about expressing the roots according to given domains. For example, solve:

$$4\cos^2 z - 12\cos x = 16 \text{ for } x \in R$$

Sol. $4\cos^2 x - 12\cos x - 16 = 0$

$$\cos^2 x - 3\cos x - 4 = 0$$

$$(\cos x - 4)(\cos x + 1) = 0$$

$$\cos x - 4 = 0 \qquad \cos x| - 1$$

$$\cos x = 4 \qquad x = \cos^{-1}(-1)$$

$$x = \emptyset \qquad x = \pi + \hat{}\pi, k \in I$$

$$\therefore \{x = \pi + 2k\pi, k \in I\}$$

Another example shows the important use of the calculator and the understanding of the role of symmetry to help determine all the roots when solving equations.

Solution: $6\sin^2 2x - 4\sin 2x + 3\sin 2x - 2 = 0$ (make equation = 0, then factor)

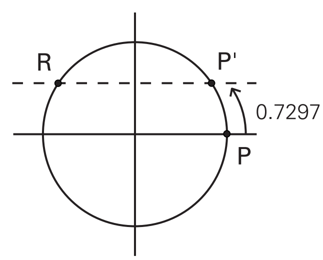
$$6\sin^2 2x - \sin 2$$

$$(3\sin 2x - 2)(2\sin 2x + 1) = 0$$

$$3\sin 2x - 2 = 0 \qquad 2\sin 2x + 1 = 0$$

$$\sin 2x = \frac{2}{3} \qquad \sin 2x = -\frac{1}{2}$$

$$2x = \sin^{-1}\left(\frac{2}{3}\right) \qquad 2x = \sin^{-1}\left(-\frac{1}{2}\right)$$



$2x$

$$2x = \begin{cases} -\frac{\pi}{6} + 2k\pi, k \in I \\ -\frac{5\pi}{6} + k\pi, k \in I \end{cases}$$

$P \rightarrow P'$ represents the arc whose sine is . But there is another arc whose sine is $\left(\frac{2}{3}\right)$ also. To find it, use symmetry: .

(dividing by 2)

$$\therefore x = \begin{cases} \frac{\pi}{12} + k\pi, k \in I \\ -\frac{5\pi}{12} + k\pi, k \in I \end{cases}$$

$\pi - .7297 \doteq 2.4119$

$$\therefore 2x \doteq \begin{cases} 0.7291 + 2k\pi, k \in I \\ 2.4119 + 2k\pi, k \in I \end{cases}$$

$$\therefore x \doteq \begin{cases} 0.365 + \\ 1.206 + \end{cases}$$

Since the domain is the final solution is:

Trigonometry 2

Worthwhile Tasks for Instruction and/or Assessment

C1/C29(Adv)/B4 *Performance*

1. The electricity supplied to a home is called "alternating current" (AC) because the current varies sinusoidally with time. The frequency (period = $\frac{1}{60}$ sec) of the sinusoid is 60 cycles per second. Suppose that at time $t = 0$ seconds, the current is at its maximum, $i = 5$ amperes, with an amplitude of 5.
   a) Draw a graph to represent several frequencies.
   b) Write an equation expressing current in terms of time.
   c) What is the current when $t = 0.01$?
   d) At what time intervals is the current at a maximum? Registering 1 amp?
   e) Use a CBL or a computer equivalent, the light probe, and a program that will gather data representing light intensity over time. (Use the program "Light 2" "Real World Math with the CBL System", if available.)
      i) Locate a simple fluorescent bulb.
      ii) Start the program.
      iii) Point the probe close to the lit fluorescent tube.
      iv) From the graph on the screen, it appears that light intensity values are rising and falling in a regular pattern. What do you think the peaks represent in terms of the tube?
      v) Calculate the period, take its reciprocal, and that is the frequency (the time required for one complete on–off cycle.
      vi) How consistent is the frequency of this model compared to the given situation (60 cycles per second)?
      vii) State the equation that represents the intensity of the fluorescent tube over time and find the times when the intensity is zero. Interpret the meaning of these values.

C29(ADV)/B4

2. Solve these equations for .
   a) $3\sin^2 x + \sin x = 0$ b)
3. Solve these equations for .
   a) b)
4. Beth's and Anna's solutions to the equation are shown. Find any errors that they have made and explain how to correct them.

Beth

$$2\cos 2x + \sqrt{2} = 0$$

$$\cos 2x = \frac{\sqrt{2}}{2}$$

$$2x = \cos^{-1}\left(\frac{\sqrt{2}}{2}\right)$$

$$2x = \frac{\pi}{4} + 2k\pi, k \in I$$

$$x = \frac{\pi}{8} + k\pi, k \in I$$

Anna

$$2\cos 2x + \sqrt{2} = 0$$

$$\cos 2x = \frac{\sqrt{2}}{2}$$

$$\cos 2x = -\frac{\sqrt{2}}{4}$$

$$x = \begin{cases} 1.932 + 2k\pi, k \in I \\ -1.932 + 2k\pi, k \in I \end{cases}$$

Suggested Resources

Brueningsen, Chris, Real-World Math with the CBL™ System, Texas Instruments, 1994