Unit 6
Applications of Trigonometry
(10 %)

Applications of Trigonometry

Outcomes

SCO: By the end of Mathematics 2204/2205, students will be expected to

B4 use the calculator correctly and efficiently

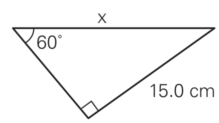
C28 analyse and solve trigonometric equations with and without technology

Elaboration – Instructional Strategies/Suggestions

Trigonometry is the study of the relationships of the measures of the sides and angles of triangles. All problems that deal with geometric situations where length or angle measure is required can be modelled with diagrams or constructions of the geometric figures.

C28 In a previous course, students developed the trigonometric ratios $\sin\theta$, , and and applied them to problems involving right triangles. Students may need to review the use of trigonometric ratios to solve problems involving right triangles.

B4/C28 Students will use their calculators correctly and efficiently for various procedures while working with trigonometric relationships. For example, when finding a missing side in a situation modeled by a right triangle, some students may set up an equation like



;

then multiply both sides by *x*: *x* sin 60° = 15.0, then divide by sin 60°, they would about have the expression: .

Using their calculator in degree mode, they would divide 15.0 by sin 60° to obtain 17.3 cm, the length of the hypotenuse. This would be efficient use of the calculator.

Students will also use trigonometric equations like the one to the right to find missing angles in right triangles.

$$\tan\theta = \frac{4.0}{7.0}$$

Some students would first change the ratio to a decimal, $\tan\theta = .57$.

Then those students would find the angle measure by using "tan $^{-1}$."

Other students may simply enter

into their calculator and solve for theta. This would be considered efficient use of the calculator.

$$\theta \doteq 30°$$

Since this chapter deals with measurement, students should be careful to properly use precision, accuracy and significant digits. Teachers may wish to review these at this time.

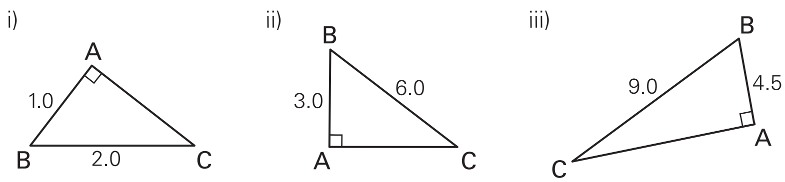
Applications of Trigonometry

Worthwhile Tasks for Instruction and/or Assessment

B4/C28

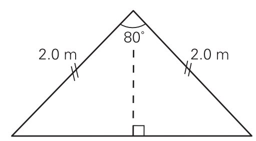
Performance

1) a) Find the measure of angle C for each of the following. From the pattern in the values, make a conjecture about a particular relationship that seems to exist in a 30º–60º–90º triangle.

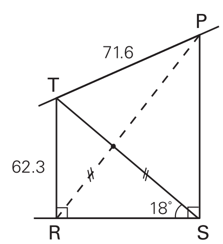


b) Test your conjecture in a situation that you create.

2) Oliver makes a shelter in the shape of an isosceles triangle. Using the given measures, find how much room above his head he will have inside the shelter. His height is 1.48 m.



3) In the town where Simon lives some roads were constructed as in the diagram. Simon needs to know how much longer the road from S to P is than the road from R to T. Can you help Simon?



4) Richard leans the top of his 7.3m ladder against the sill of a window that is 6.5m above the ground.
   a) At what angle to the ground will his ladder be?
   b) If the ladder is rotated, without moving its feet, to lean in the opposite direction against a building that is 5.1 m away from the first building, at what angle will it be to the ground?

Suggested Resources

Brueningsen, Chris et al, *Real-World Math with the CBL™ System*, Texas Instruments

Meiring, Steven P., "A Core Curriculum," *Addenda Series 9-12*, NCTM, 1992

"Geometer's Sketchpad", Key Curriculum Press

Applications of Trigonometry - Area of a Triangle

Outcomes

SCO: By the end of Mathematics 2204/2205, students will be expected to

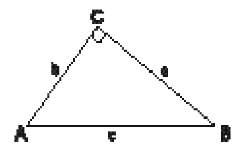
B6 derive and analyse the Law of Sines, the Law of Cosines, and the formulas 'area of a triangle

'

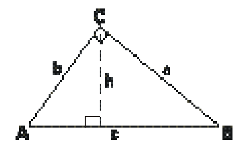
D5 apply the Law of Sines, the Law of Cosines, and the formulas ' area of a triangle $ABC = \frac{1}{2}bc\sin A$ ' to solve problems

Elaboration – Instructional Strategies/Suggestions

B6 Sometimes finding the area of any triangle can be done more efficiently by using the area formula, 'area of a triangle' $= \frac{1}{2}bc\sin A$. To develop this formula, students should be asked to write how they would find the area of triangle ABC.



Students would write $A = \frac{1}{2}$ base $\times$ height .



In this triangle, the base is c, so $A = \frac{1}{2}ch$

The teacher would then ask students to replace the 'h' with an expression using sin A.

Students would write

or,

so, filling into A = $\frac{1}{2}$ base $\times$height

or without brackets

D5 Students should apply this formula in various problem - solving situations involving area. To use this formula to find area students should realize that they need any two sides and the included angle measure of any triangular shape. When the area of a triangular shape is given, students can use this formula to find any of the missing three measures in the formula, if the other two are given. For example, if the area of a triangular region on a stage was to be carpeted with 37 square metres of carpet, and two adjacent sides measured 12.0 m and 6.7 m, then the angle between these sides could be found:

$$\text{Area} = \frac{1}{2}bc\sin A$$

$$37 = \frac{1}{2}(12.0)(6.7)\sin A$$

$$A \doteq 67°$$

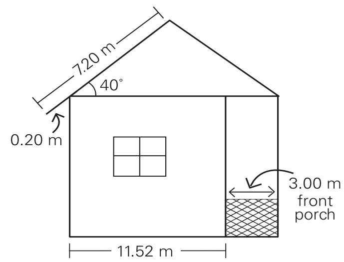
Applications of Trigonometry

Worthwhile Tasks for Instruction and/or Assessment

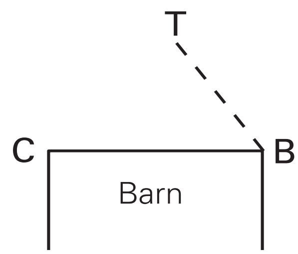
D5

Performance

1) Hilary wants to paint the triangular gable ends of her log cabin. She knows that a can of paint will cover 39 m^2. She expects to have to paint two coats. If a can costs $29.95, how much money will she have to spend?



2) Cousin Barney is building a new corral on the side of his barn for his new lamb, Huey. He measures the barn length BC to be 15.25 m. There is already a fence from one end of the barn to a tree (T) with a length 21.62 m. Barney has just spread seed that covers all 120.50 m^2 inside the triangular region C–B–T. How long will the fence be that goes from C to T?



3) The area of a triangle is 100cm^2. Two sides are known such that one is 3 times the length of the other. The angle between them is 47°. What is the measure of all the sides?

B6

Journal

4) Ask students to explain why c sin A from the formula

Area of a triangle $= \frac{1}{2} bc \sin A$ is the same as 'h' in the formula

Area of a triangle $= \frac{bh}{2}$.

Suggested Resources

Meiring, Steven P., "A Core Curriculum," *Addenda Series 9-12*, NCTM, 1992

Applications of Trigonometry

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D3 apply sine and cosine ratios and functions to situations involving non-acute angles

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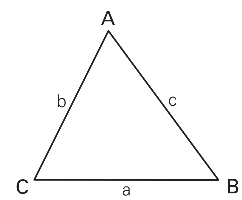
Elaboration – Instructional Strategies/Suggestions

B6 To develop the Law of Sines, students might begin like this (or they might use a geometry software package to explore in the same way):

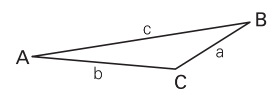
Questions 1–3 refer to the acute ΔABC on the right.



1) Measure each side to the nearest tenth of a centimetre.
   a) c b) a c) b
2) Measure each angle to the nearest degree.
   a) $\angle A$ b) c)
3) Calculate each of the following to 1 decimal place.

   a) b) c)

4) Repeat questions 1–3 for the obtuse ΔABC.
5) Draw an acute triangle, ΔABC of your own and repeat questions 1–3.

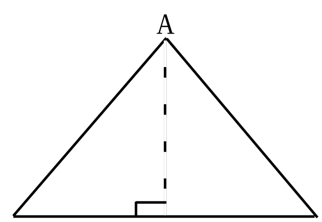


6) Draw an obtuse triangle, ΔABC of your own and repeat questions 1–3.
7) Based on the results of question 3, what can you conjecture about the relationship

   between ?

Students should conclude that these ratios are equal, and they should understand that they should be able to prove that they are equal.

To help students with this proof, teachers might ask some students to write a trigonometric equation using sin B that shows the height AD, and others to use sin C.



Students would respond with

so AD = c sin B

and others with: so AD = b sin C

Teachers and students would reason that the conclusion would have to be

c sin B = b sin C since they are both equal to AD.

Students should then verify that is the same as c sinB = b sin C.

(Multiply both sides of the equation by sin C sin B—the lowest common denominator or cross multiply).

D5 Students should apply the Law of Sines to solve various problems.

... continued

Applications of Trigonometry

Worthwhile Tasks for Instruction and/or Assessment

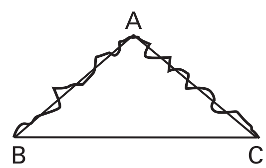
B6

Performance

1. (Z-level only) Draw any triangle ABC, and the altitude from B. Use this to derive the Law of Sines.
2. (Y-level only) Draw any triangle ABC, and the altitude from A. Use this to derive the Law of Sines. Hint: Express the length of the altitude in terms of the sine of an angle, and again as the sine of a different angle.
3. If $\frac{\sin A}{a}=\frac{\sin B}{b}$ how do you know for sure that $\frac{\sin C}{c}=\frac{\sin B}{b}$? Explain.

B6/D5

4. What information must be given in a problem that would lead you to use the Law of Sines?
5. a) While visiting a ski lodge in Switzerland, Real noticed the profile of a mountain in the distance, and how triangular it appeared. He estimated the angle measure at C to be about 30°, and B to be about 40°. The sign advertises a T-bar ride up the s lope AB, a distance of 2917 m. Show that the ski run down the side AC must be at least 3750 m.



6. a) Make up a problem where a side length can be determined using Law of Sines. Show the solution.

   b) Make up a problem where an angle measure can be determined using Law of Sines.

Suggested Resources

Applications of Trigonometry - Law of Sines

Outcomes

SCO: By the end of Mathematics 2204/2205, students will be expected to

D5 apply the Law of Sines, the Law of Cosines, and the formula 'area of a triangle $ABC = \frac{1}{2}bc \sin A$' to solve problems

Elaboration – Instructional Strategies/Suggestions

D5 Students will use the Law of Sines to find a missing side length when two angle measures and one side length are given. Or, they will use the Law whether they have one angle already and are looking for another, given the opposite sides. Any one proportional statement from the formula called the Law of Sines is all that is used at one time.

$$\frac{a}{\sin A} = \frac{b}{\sin B} \text{ or } \frac{b}{\sin B} = \frac{c}{\sin C}$$

Students should realize that some students may use the formula in this form:

$$\frac{\sin A}{a} = \frac{\sin C}{c}$$

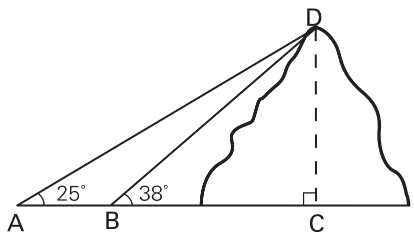
Applications of Trigonometry

Worthwhile Tasks for Instruction and/or Assessment

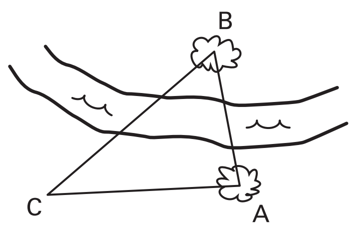
D3/D5

Performance

1. Surveyors cannot get to the inside centre of a mountain easily. Therefore, a mountain's height must be measured in a more indirect way. Find the height of the mountain if AB = 600m.



2. Surveyor Sally had to determine the distance between two large trees situated on opposite sides of a river. She placed a stake at C, 100.0 m from point A. Sally then determined ACB to be 45º and ∠CAB to be 80º .



   a) Ask students to help her find the distance between the trees.

$$\frac{\sin A}{a} = \frac{\sin B}{b}$$

   b) Ask students to create a different problem, using the above diagram.

Journal

3. Billy is using to solve a problem. Billy's dad said that in his day he would have used $\frac{a}{\sin A} = \frac{b}{\sin B}$. Is Billy or his dad correct? Explain.

Suggested Resources

Meiring, Steven P., "A Core Curriculum," *Addenda Series 9-12*, NCTM, 1992

"Geometer's Sketchpad" Key Curriculum Press

Applications of Trigonometry - Law of Sines

Outcomes

SCO: By the end of Mathematics 2204/2205, students will be expected to

D3 apply sine and cosine ratios and functions to situations involving non-acute angles

D5 apply the Law of Sines, the Law of Cosines, and the formula 'area of a triangle

' to solve

problems

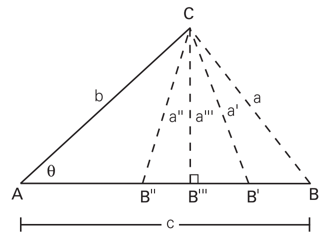
B4 use the calculator correctly and efficiently

Elaboration – Instructional Strategies/Suggestions

D3/D5/B4 When applying the Law of Sines, students must consider the possibility that there may be more than one triangle with certain given measurements. One will lead to an angle measure greater than 90°, but the calculator will give the corresponding supplementary measurement. They should consider what happens given the following criteria:

When given a measure for $\angle A$, a length 'b' and a length 'a', the points A and C are fixed, but the point B is not. The side CB could be positioned in any of the locations indicated by the dotted line, depending on how long CB, (length 'a') is and the measure of the

.



The shortest 'a' can be is the perpendicular length from C to AB''' and is determined by b sin A. If 'a' is a little longer, then B might be in location B' or B''. This will be true until a exceeds b, B can be the only location where length 'a' can rest on AB.

Thus:

1. if a = b sinA, then there is only one triangle and it is a right triangle with one .
2. if a > b sin A, but a < b, then two triangles are possible with B at B'or B''.
3. if a > b sinA and a > b, then there is only one with B at location B.

Situation (2) is called the 'ambiguous' case, and students should be given the opportunity to develop this through an activity.

Applications of Trigonometry

Worthwhile Tasks for Instruction and/or Assessment

D5/D3/B4

Performance

1. Sketch a diagram, including some measurements that would ask students to solve for a particular side in a given triangle. Have them show how the diagram could be drawn differently that results in a different but correct answer for an angle. Explain how the students would know that these different answers might both be correct answers.
2. There was a shipwreck. As a Coast Guard vessel approached the wreck, a rescue helicopter was located at an angle of elevation of 35^o and 380 km from the Coast Guard vessel. The helicopter dropped its rescue rope. The wind was blowing quite hard, so the rope had to be let out further (25m altogether) in order to reach the wreck. How far was the wreck from the Coast Guard vessel if the vessel, wreck, and helicopter were in the same plane?
3. Matthew enjoys swimming in the ocean. One day Matthew decided to swim 9.2km from island A to island B, then after resting, he swam 8.6km to island C. If island C to island A to island B forms a 52^o angle, determine how far Matthew has to swim to return to island A.

Suggested Resources

Applications of Trigonometry

Outcomes

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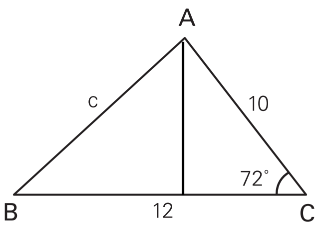
D3 apply sine and cosine ratios and functions to situations involving non-acute angles

D5 apply the Law of Sines, the Law of Cosines, and the formula 'area of a triangle ' to solve problems

Elaboration – Instructional Strategies/Suggestions

B6/D3 Students will interact with the teacher to develop a procedure for obtaining measurements of triangles that are not right-angled (oblique triangles). They should develop through teacher-led discussion, or a directed activity, the Law of Cosines and the Law of Sines.

Teachers should help students discover or develop the Law of Cosines which, stated symbolically, in one form, is $c^2 = a^2 + b^2 - 2ab \cos C$. Students should also be able to state this Law, beginning with a^2 or b^2.



D3/D5 Applied to the given diagram to find AB, the Law of Cosines would be written as:

$AB^2 = BC^2 + AC^2 - 2(BC)(AC) \cos C$

B6 To develop this formula, students might begin by dropping a perpendicular from A to $\overline{BC}$ at point R. This creates some right triangles and splits into two lengths x and 12 – x. Encourage students to use right triangle trigonometry and the Pythagorean Theorem to express certain lengths and angle measures in terms of BC, AC, and the cosine of .

The activity might begin by asking students to express c^2 using the Pythagorean Theorem. Students should notice that

$c^2 = AR^2 + BR^2$

BC = BR + RC and try to use the value for BC in the formula. Since . . .

BC = 12, let RC = x and BR = 12 – x

Now, substituting into and explaining BR^2

$c^2 = AR^2 + (12 - x)^2$
$= AR^2 + 12^2 - 2(12)x + x^2$

rearranging gives ...

$AR^2 + x^2 = AC^2$, and BC = 12 ...

RC, or x can be expressed (using trig) as . . .

$10 \cos 72^0$

$AC \cos C = RC$

AC = 10 and C = 72°, so . . .

substituting into ...

$c^2 = AC^2 + BC^2 - 2(BC)(AC)\cos C$

This representation now allows students to find AB ('C') using the given measurements, and without having to have right angles.

D5 Students should apply the Law of Cosines in various situations. Students should think about using this Law when they have a situation where there is no right angle, and they need a third side length when the other two sides of the triangle are given, or when the and side information does not lead easily to the Law of Sines.

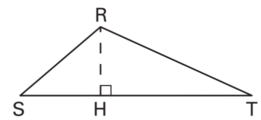
Applications of Trigonometry

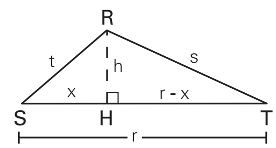
Worthwhile Tasks for Instruction and/or Assessment

B6

Performance

1. (Z-level only) Using the diagram given, derive the Law of Cosines.





2. (Y-level only) Given: ΔRST and altitude RH.
   Prove: $s^2 = t^2 + r^2 - 2rt \cos S$
   Hint:
   i) Use Pythagorean Theorem to express t^2 and s^2.
   ii) Express both of the above beginning with "$h^2 =$".
   iii) What new equation can be formed now?
   iv) Expand binomials if possible.
   v) What you're trying to prove has s, t, and r as the variables. Replace other variables in terms of s, t, and r.

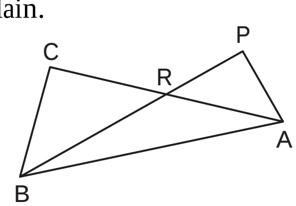
B6/D5

3. Explain how you would know when to use the Law of Cosines in a problem.
4. In class, Marlene said that she could find AB in question 2(p. 183), using the Law of Cosines. Is she correct or not? Explain.

D5/D3

5. a) Given

ΔBPA
$BC = 87.4$ m
$CA = 101.5$ m
$PA = 56.2$ m
$\angle C = 112.4°$
$\angle PAC = 37.2°$



find the area
   b) Create a problem context for this given information.
   c) Create another problem like this, where you must first obtain a side, then determine an area.
6. What is the measure of the largest angle in the triangle with side lengths 16, 21, and 24cm?

Suggested Resources

Applications of Trigonometry - Law of Cosines

Outcomes

SCO: By the end of Mathematics 2204/2205, students will be expected to

D3 apply sine and cosine ratios and functions to situations involving non-acute angles

D5 apply the Law of Sines, the Law of Cosines, and the formula 'area of a triangle $ABC = \frac{1}{2}bc \sin A$' to solve problems

Elaboration – Instructional Strategies/Suggestions

D3/D5 Usually the formula for the Law of Cosines is used to find a missing side, given the other two sides and the angle between the given sides (the included angle). The formula can be re-stated as

$b^2 = a^2 + c^2 - 2ac \cos B$, or
$a^2 = b^2 + c^2 - 2bc \cos A$ to better set up the solution process.

The formula is structured so that students need not memorize it. Instead, if the measure 'a' is needed, then the other side of the formula is made up of all 'b's and 'c's. If the measure 'b' is needed, then the other side of the formula is all 'a's and 'b's. The angle used in the formula is always the angle opposite the side needed.

Sometimes students will use the formula to find an angle measure when all 3 side lengths are given. They don't need to rearrange the formula first. For example, given a = 12.0 cm, b = 9.5 cm, and c = 7.2 cm, students could find any of the angle measures. If they wanted $\angle A$, they would begin the formula with a^2:

$$a^2 = b^2 + c^2 - 2bc \cos A$$
$$12.0^2 = 9.5^2 + 7.2^2 - 2(9.5)(7.2) \cos A$$
$$144.0 = 90.25 + 51.84 - 136.8 \cos A$$

Next, they need to isolate the variable term:

$$144.0 - 90.25 - 51.84 = -136.8 \cos A$$

then isolate the variable:

$$-0.01396 = \cos A$$
$$A \doteq \cos^{-1}(-0.01396)$$
$$A \doteq 90.8°$$

Students would conclude that the angle A measures about 91°. Students need to think about using the Law of Cosines when they have a situation where there is no right angle and they need a third side length, given the other two, or an angle measure, given all the side lengths.

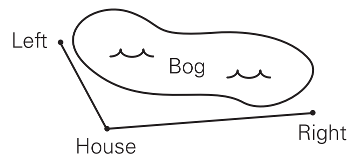
Applications of Trigonometry

Worthwhile Tasks for Instruction and/or Assessment

D3/D5

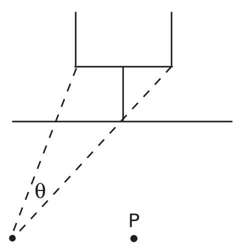
Performance

1. A farmer wants to find the length of the back of his property, which is mostly bog land. He knows that his left and right boundary lines connect near his house at an angle of 147°. The left boundary length is 90 m and the right is 110 m. Help the farmer.



2. Terry is building an A-frame cabin in the woods. The length of each of two rafters is 8.50m. If the angle of the apex of the frame is to be 46°, calculate the proposed width of the cabin at the base.

3. a) A football player is attempting a field goal. His position on the field is such that the ball is 7.5m to the left upright of the goal post and 10.0m to the right up-right of the goal post as shown. The goal posts are 4.3m apart. Find the angle marked .



   b) If the ball is moved to the middle of the field, position P, then the ball is equidistant to both uprights, approximately 8.5 m each. Find the angle corresponding to from

θ

   this position.

D3/D5

Journal

4. a) Explain why the Law of Cosines might be a useful relationship to try to remember.
   b) How can you help yourself remember it? (Hint: It is the Pythagorean Theorem, plus or minus some adjustment factor. How can I remember the adjustment factor?)

Suggested Resources

Brueningsen, Chris et al, *Real-World Math with the CBL™ System*, Texas Instruments

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D5 apply the Law of Sines, the Law of Cosines, and the formula 'area of a triangle ' to solve problems

Elaboration – Instructional Strategies/Suggestions

D3/D5

Teachers may want to discuss with students that the Law of Cosines 'looks' like the Pythagorean Theorem with an adjustment factor to make up for the lack of a right angle.

$$c^2 = a^2 + b^2 - \text{adjustment factor}$$

They may want to have students examine several situations where the measure of $\angle C$ has various values, say 40°, 50°, 70°, 85°, 90°, 95°, 110°, 130°, ... Students would evaluate the 'adjustment factor' 2ab cos C and note that as m ,
. As m gets larger and larger beyond 90°, 2ab cos C takes a 'larger' negative value.

Talk to students now about:

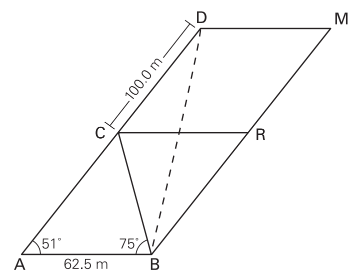
if , then (Pythagorean Theorem)

if , then c should get smaller. $c^2 = a^2 + b^2$ - (positive adjustment factor)

if , then c should get larger. $c^2 = a^2 + b^2$ - (a larger and larger negative value.)

Students should be prepared to combine both the Laws of Sines and Cosines in the same question when required.

For example, farmer Jones' property (ABRC) is a parallelogram (see diagram). He is given more land (DMRC). He needs to know the distance from D to B. He would first have to use the Law of Sines to get BC, then the Law of Cosines to BD.



Applications of Trigonometry

Worthwhile Tasks for Instruction and/or Assessment

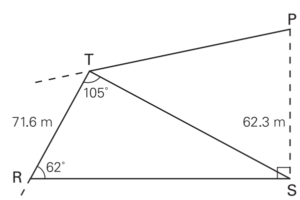
D3/D5

Activity

1. Given with b = 4.0 cm and c = 3.0 cm:
   a) Ask students to determine ‘a’ if .
   b) Ask students to evaluate 2bc cos A, if .
   c) Ask students to determine ‘a’ again using $a^2 = b^2 + c^2 – 2bc \cos A$, given . Make a conjecture about this new formula.
   d) Ask students to construct an accurate diagram of ΔABC with b = 4.0 cm, c = 3.0 cm, and .
   e) Ask students to
      i) predict if a < 5.0 cm, a = 5.0 cm, or a > 5.0 cm and explain their choice.
      ii) measure with a ruler the length ‘a’ to the nearest 10th of a cm and record it.
      iii) calculate the length ‘a’ using $a^2 = b^2 + c^2 – 2bc \cos 80^o$.
   f) Ask students to repeat step (e) given .
   g) Ask students to describe the pattern and how it fits with their conjecture in (c).
   h) Ask students to repeat step (e) given $m\angle A = 91°, 95°, 115°, 120°$
   i) Ask students to describe the pattern and how it fits with their conjecture in (c).
   j) Ask students to make a statement about how the formula $a^2 = b^2 + c^2 – 2bc \cos A$ can be used.
   k) Glen conjectured that if he knew the side measures, he could determine the using this new formula. Do you agree or disagree? Explain.
   l) Colin conjectured that, if he knew the length of ‘b’ and ‘a’ and the , he could determine the length C. Explain what Colin must be thinking.

ΔABC m∠A = 90° m∠A = 80°, 65°, 85°, 89° m∠C

2. In the town where Simon lives, some roads were constructed as in the diagram. Simon needs to know how much longer the road from P to T is than the road from R to T.



Suggested Resources

Brueningsen, Chris et al, *Real-World Math with the CBL™ System*, Texas Instruments

Meiring, Steven P., “A Core Curriculum,” *Addenda Series 9-12*, NCTM, 1992