Unit IV - Simplifying Algebraic Expressions and Rearranging Formulas (25%)

Suggested Instruction and Assessment Time for this Unit:
12-13 classes(Assumes 55-60 minute classes)

Outcomes

SCO: By the end of Mathematics 3103 students will be expected to:

B13 add, subtract, multiply and divide rational expressions

B14 use the four basic operations to simplify complex fractions

Elaboration - Instructional Strategies/Suggestions

B13/14

Students are expected to find the LCD of two or more rational expressions and be able to apply it in the context of adding or subtracting. It is not expected that all four operations be done at the same time, but teachers may wish to do so in order that students will have the opportunity to compare and contrast more readily the methods used for each operation.

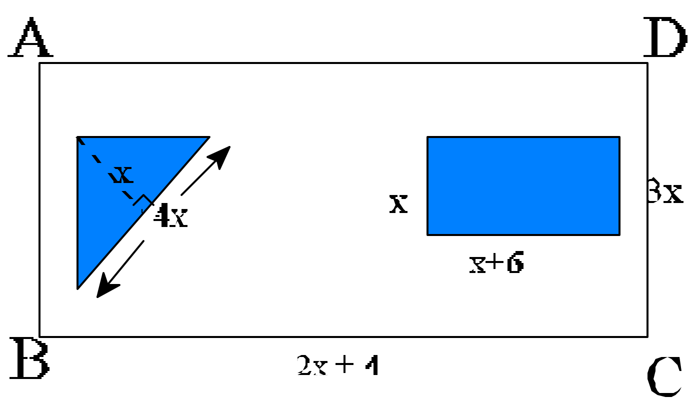
These outcomes will provide students with an opportunity to practice skills related to factoring as well as an opportunity to re-visit the order of operations in a different context.

There should be a focus on mental math as well as pencil and paper skills. For example, students should be able to simplify problems mentally such as the ones which follow:

$$\frac{(2x-3)(x+5)}{(x+5)^2} \times \frac{x+5}{(2x-3)}$$

Students should solve problems requiring multiple steps to completion. For example:

- What percentage of the rectangle ABCD is shaded?



Worthwhile Tasks for Instruction and/or Assessment

Mental Math

B13.1 a) Simplify: b) Simplify: $\frac{2x+3}{3+2x}$

B13.2 What is the LCD for a) $\frac{2}{5t}$ and $\frac{t+1}{t^2}$

b) and ?

B13.3 Add and simplify:

B13.4 Simplify: $\frac{(2x-5)(x+1)}{(x+1)^2} \times \frac{x+1}{(2x-5)}$

B13.5 Simplify: $\frac{xy^2}{3x} \div \frac{y^2}{6}$

B14.1 Simplify:

Pencil and Paper

B13.6 Reduce to lowest terms:

a) b)

c) $\frac{6x^2 - 13x + 2}{36x^2 - 6x}$ d)

B13.7 Simplify:

a) b)

c) $\frac{3x^3 - 2x^2}{3x^2 + 10x - 8} \div \frac{x^3 - x^2 - 2x}{x^2 + 8x + 16}$

Resources

Mathematical Modeling, Book 2 (Math 2204 text)
p.152, Focus D
p.153, 154 (except #8(g) & (h))

Using Advanced Algebra, P.115-135

Mathematics 11 (National Edition), p.85-104

Mathematics 11: Principles & Process, p.78 - 91

Math In Context 9,
p.356-365
p.542-543 (sec. 11.9-11.12)

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B14 use the four basic operations to simplify complex fractions

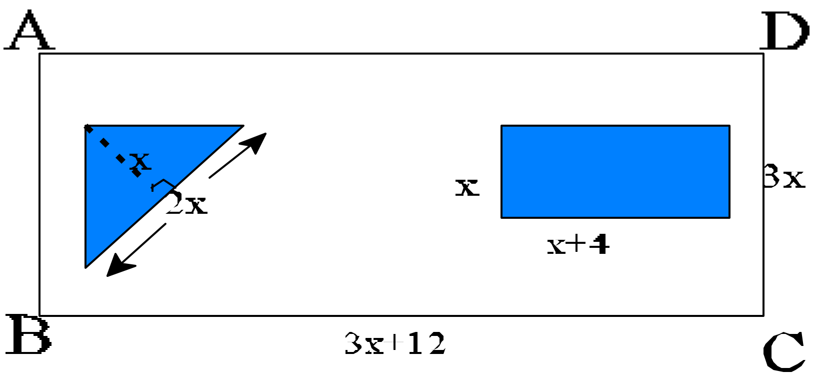
Elaboration - Instructional Strategies/Suggestions

Worthwhile Tasks for Instruction and/or Assessment

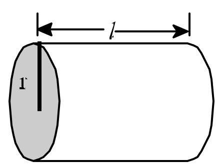
B13.8 Add or subtract as indicated and reduce your answers when possible:

a) b)

B13.9 What percentage of rectangle ABCD shown below is shaded?



B13.10 A bar of iron in the shape of a cylinder is to be nickel plated on its curved surface, with the ends being left untreated. Show that the fraction of the total surface area to be plated is $\frac{l}{r+l}$.



(**Note**: Area of a circle is πr^2. To get the area of the curved surface, try “unwrapping” the cylinder by considering how a rolled up sheet of rectangular paper can be used to form a cylinder. This rectangle has the same area as the curved surface. How can you use the circle on the end of the cylinder to find an expression for the length of this rectangle?)

B14.2 Simplify each complex fraction:

a) b) c)

Resources

Access (Alg) LB 1-88; LC 1-92; LD 1-68; LE 1-212; LF 1-128; LG 1-108; LH 1-120; LI 1-132; LJ 1-136

Outcomes

SCO: By the end of Mathematics 3103 students will be expected to:

C7 solve equations involving rational expressions and determine extraneous roots

Elaboration - Instructional Strategies/Suggestions

Students will need to be reminded that division by 0 is impossible. Teachers may wish to discuss the reason for this by reviewing the meaning of division. One way to do this is to ask, "What is 10 ÷ 2?" When students respond, "5", ask them *why* this is the answer. Let them think for a moment and discuss possible answers. If nobody provides an acceptable reason, remind them that this problem really asks them "How many 2's do you have to add to get 10?". or "How many sets of 2 would you subtract 2 from 10 before all 10 are used?" Ask them to think about this for other similar questions, then ask, "What is 2 ÷ 0?" i.e. " How many 0's do you have to add to get 2?" They can then think about dividing any number by 0 and why division by 0 is considered to be undefined.

When considering rational expressions, students should first consider the restrictions on the replacement set (domain) of the variable e.g.

$$\frac{x+2}{x-3}+\frac{x+3}{x-2}=\frac{5}{x^2-5x+6}$$

They should first factor, if possible, the denominators. Next, they should consider which value(s) of *x* would make any of the denominators equal 0 - these will have to be excluded from the replacement set. If any of these values occur in a final solution to the problem, they will be considered extraneous. Students must realize that even when they solve an equation correctly, it is still possible to get answers that cannot work in the context of the original problem. There may be in fact no answer at all!

There are two approaches that teachers may wish to use with students, but both involve the use of the LCD. The above problem is solved by both methods below:

<u>Method 1</u>

Restrictions: $x \neq 3, 2$

Multiply each rational expression by the LCD

$$\frac{(x-3)(x-2)(x+2)}{x-3}+\frac{(x-3)(x-2)(x+3)}{x-2}=\frac{5(x-3)(x-2)}{(x-3)(x-2)}$$

Reduce and multiply left over factors:

$(x - 2)(x + 2) + (x - 3)(x + 3) = 5$

$x^2 - 4 + x^2 - 9 = 5$

Simplify and solve the resulting equation:

$2x^2 - 13 = 5$

x = 3 or -3 but x = 3 is extraneous due to our restrictions, so x = -3 is the

Worthwhile Tasks for Instruction and/or Assessment

Mental Math

C7.1 What value(s) must be excluded from the replacement set of the rational expression ?

C7.2 What is the LCD for ?

C7.3 Add and simplify:

C7.4 What is the LCD for and ?

Paper and Pencil

C7.5 Solve each of the following equations:

a)

b)

c)

d)

e)

f) $\frac{2x}{2x+3}+\frac{9x}{6x^2-x-15}=\frac{15}{6x^2-x-15}$

Enrichment: Solve

Resources

Using Advanced Algebra, p.114-117, 136-139

Advanced Mathematics, Brown, Richard and Robbins, David. p.55

Access (alg) LD 1-68, LK 1-136, LL 1-100

Further work is necessary for factoring trinomials of the form $ax^2 + bx + c$

Outcomes

SCO: By the end of Mathematics 3103 students will be expected to:

C7 solve equations involving rational expressions and determine extraneous roots

Elaboration - Instructional Strategies/Suggestions

Method 2 Same restrictions as in method 1, but this time rewrite each rational expression using the LCD:

$$\frac{(x+2)(x-2)}{(x-3)(x-2)}+\frac{(x+3)(x-3)}{(x-2)(x-3)}=\frac{5}{(x-3)(x-2)}$$

Multiply the factors in the numerators:

$$\frac{x^2-4}{(x-3)(x-2)}+\frac{x^2-9}{(x-2)(x-3)}=\frac{5}{(x-3)(x-2)}$$

Add:

$$\frac{2x^2-13}{(x-2)(x-3)}=\frac{5}{(x-3)(x-2)}$$

Because the denominators are the same, the numerators must now be equal, thus leaving the same equation as in method 1:

$2x^2 - 13 = 5$

Note: As a shortcut, the denominators in the first line of this method could have been eliminated.

Worthwhile Tasks for Instruction and/or Assessment

Resources

Outcomes

SCO: By the end of Mathematics 3103 students will be expected to:

B15 rearrange formulas to solve for any variable

Elaboration - Instructional Strategies/Suggestions

B15 Teachers should allocate a maximum of two classes for this outcome. This outcome is partially intended to help students prepare for outcome B16. Teachers may wish therefore to place a special emphasis on rearranging the formulas they intend to use with outcome B16.

While students should be exposed to a variety of formulas, the intent is to develop the *skill* of rearranging formulas. It is not necessary, therefore, to have students rearrange all formulas suggested on the next several pages. Have students practice with enough formulas to become reasonably comfortable, especially since they will have ample opportunity to practice again in the next outcome. Indeed, in small classes with able students, teachers may wish to cover B15 and B16 concurrently.

Formulas to be rearranged should include the operations of multiplication, division, addition, subtraction, squaring, and square root. Some formulas might also require factoring in order to rearrange, for example:

Solve for P: $P + Prt = I$

$P(1 + rt) = I$

$P = I / (1+rt)$

It would be wise for students to construct a formula sheet containing all formulas that will be used in the next outcome, B16. Particular emphasis should be placed on area, surface area, and volume formulas.

Worthwhile Tasks for Instruction and/or Assessment

Some of the formulas teachers may wish to have students rearrange are:

Area of a triangle: $A = \frac{1}{2}bh$

Area of a circle: $A = \pi r^2$

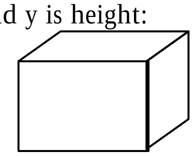
Circumference of a circle: $C = 2\pi r$

Speed (from Physics): $v = d/t$, where v = speed, d = distance, t = time

Surface area of a box with a square base, where x is width and y is height:

With top: $S = 2x^2 + 4xy$

Without Top: $S = x^2 + 4xy$



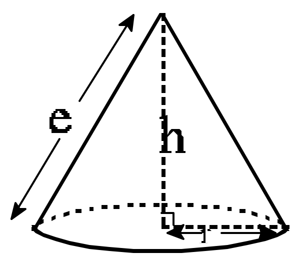
Surface area of a cone:

Area of circular base is πr^2

Area of curved surface is πre

Total surface area is $S = \pi r^2 + \pi re$, where *e* is the slant height.

(e.g. given the surface area of a cone is $30cm^2$, solve for e)

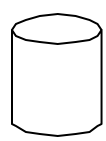


Volume of a cone: $V = \frac{1}{3}\pi r^2 h$, where *r* is the radius of the base, *h* is the height

Surface area of a cylinder:

S = area of curved surface + area of top and bottom

$= 2\pi rh + 2\pi r^2$ Where r is the radius of the base and h is the height.



Note: For the area of the curved surface, it is useful for students to see that a rectangular piece of paper can be formed into a cylinder by joining the ends. Students should then be able to see that the length of the rectangle is the same as the circumference of the base of the cylinder, which is $2\pi r$. The width of the rectangle is the same as the height of the cylinder.

Resources

Mathematics 10 (Revised Edition)
p.130 - 134

Outcomes

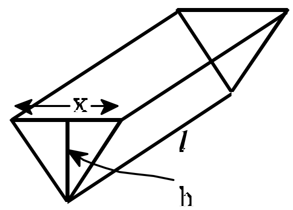
SCO: By the end of Mathematics 3103 students will be expected to:

B15 rearrange formulas to solve for any variable

Elaboration - Instructional Strategies/Suggestions

Worthwhile Tasks for Instruction and/or Assessment

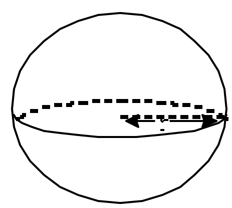
Volume of a solid with constant cross-sectional area: V = Bh, where B is the area of the base. In the case of a cylinder, for example, the cross-section is circular so πr^2 would be substituted for B in the formula.



For the triangular prism at right, students would find the area of the triangular base B = ½ xh to get the formula for the volume to be V = ½xh*l*

Volume of a sphere: $V = \frac{4}{3}\pi r^3$. Solve for *r*.

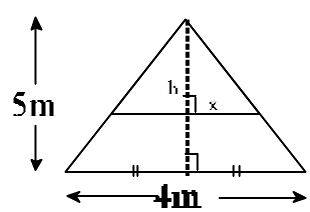
Surface area of a sphere: $S = 4\pi r^2$ (e.g. given surface area of a certain sphere is 100π cm^2, solve for r)



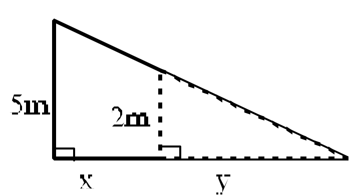
Pythagorean Theorem: $c^2 = a^2 + b^2$ (Solve for a, b, or c)

Similar triangles:

ex. 1 Students would be expected to write the relationship $\frac{h}{5} = \frac{x}{2}$ and solve for x or h.



ex.2 Express x as a function of y.



Linear functions e.g. y = mx + b. Solve for m, x, or b

Simple rational functions e.g. Solve for x: $y = \frac{2x}{x-2}$

Resources

Outcomes

SCO: By the end of Mathematics 3103 students will be expected to:

B15 rearrange formulas to solve for any variable

Elaboration - Instructional Strategies/Suggestions

Worthwhile Tasks for Instruction and/or Assessment

Sample Questions

B15.1 Solve the formula for volume of a cone

a) for *h* and

i. use it to find *h* if the volume is 100 cm^3 and the $r = 5$ cm.

ii. predict the height if the volume stays the same for a radius of 10 cm and for a radius of 2 cm.

b) for *r* and

i. use it to find the radius if the volume is 100 cm^3 and the height is 8 cm.

ii. predict the radius if the volume stays the same and the height changes to 2 cm.

B15.2 Solve the formula for surface area of a cylinder

a) for *h* and

i. use it to find the h if the surface area is 100 cm^2 and the $r = 3$ cm.

ii. predict the height if the surface area stays the same for a radius of 1.5 cm and for a radius of 2 cm.

b) For *r* and

i. use it to find the radius if the surface area is 100 cm^3 and the height is 4 cm.

ii. predict the radius if the surface area stays the same and the height changes to 2 cm.

Resources

Outcomes

SCO: By the end of Mathematics 3103 students will be expected to:

B16 simplify problem situations involving functions of two or more variables to functions of one variable

Elaboration - Instructional Strategies/Suggestions

B16 In mathematics we most often deal with functions of one variable. However, in practical applications functions of two or more variables are often needed. Students should understand that in some situations these functions can be reduced to functions of one variable using a substitution. The following might be a good place to start discussion.

- The length of a rectangle is twice the width. Express the area of the rectangle as a function of

  a) the width

  Solution: Since $A = lw$ and $l = 2w$, then $A = (2w)w$, so $A = 2w^2$. The original area formula was a function of two variables, or $A(l,w) = lw$. The final function is now only a function of one variable, or $A(w) = 2w^2$.

  b) the length

  Solution: Since $l = 2w$, then $w = l/2$ and so $A(l,w) = lw$ can be expressed as $A(l) = l(l/2) = l^2/2$

These problems generally involve rearranging one formula or expression before substituting into another. For example,

- A box with a square base and no top requires 5m^2 of material to construct. If the width is x and the height is y, express the volume of the box as a function of its width.

Solution: $S = 4xy + x^2$ is the formula for surface area, and S is 5m^2. Therefore, we can substitute and rearrange to solve for y as follows:

$5 = 4xy + x^2$

$5 - x^2 = 4xy$

$$\frac{5-x^2}{4x} = y$$

Since $V = x^2y$, then substituting for y gives

$$V = x^2\left(\frac{5-x^2}{4x}\right)$$

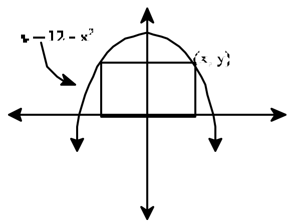
$$= x\left(\frac{5-x^2}{4}\right)$$

$$= \left(\frac{5x-x^3}{4}\right)$$

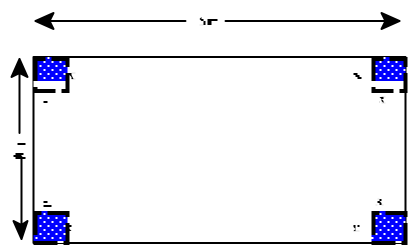
Worthwhile Tasks for Instruction and/or Assessment

(Note: See Solutions provided after the questions)

B16.1 A rectangle has vertices on the x-axis and on the parabola as shown. Express the area of the rectangle as a function of x.



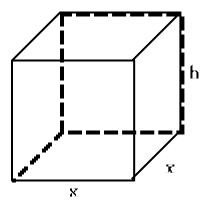
B16.2 An 8 m by 15 m sheet of metal has congruent squares cut from the corners and the edges are then turned up to make an open box with no top. If x is the side length of a square cut from each corner, express the volume of the box as a function of x.



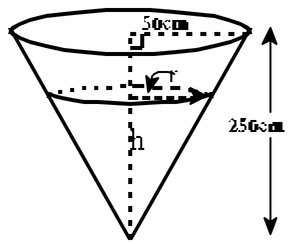
B16.3 A manufacturer produces metal boxes that have a square base, no top, and a volume of 4000 cm^3. Express the surface area of the box as a function of

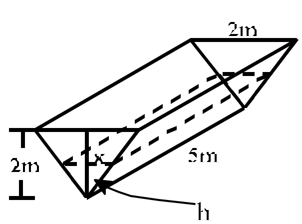
a) x

b) h



B16.4 Find the volume of water in the conical tank as a function of the height, h, of the water.





Resources

Advanced Mathematics, Brown & Robbins. p.132 - 139 (Parts of the problems on p.83 and p.590-592 can also be used if the portions on maximize and minimize are eliminated). The teacher's handbook includes detailed solutions.

In addition, most beginning calculus texts include material on *Extreme Value Problems*. Parts of these problems can be used as well.

Outcomes

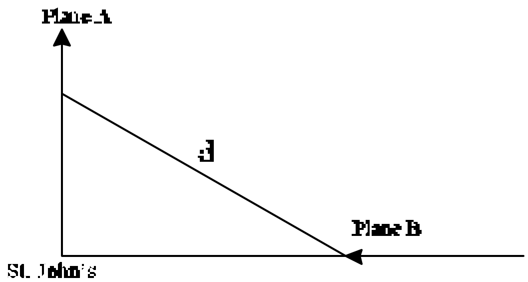
SCO: By the end of Mathematics 3103 students will be expected to:

B16 simplify problem situations involving functions of two or more variables to functions of one variable

Elaboration - Instructional Strategies/Suggestions

Worthwhile Tasks for Instruction and/or Assessment

B16.6 Plane A leaves St. John's at noon and travels north at a rate of 300km/h. At the same time, plane B is 800km east of St. John's and traveling west at a rate of 250km/h. Express the distance, d, between Plane A and Plane B as a function of the number of hours, *t*, after noon, and then use this function to find the distance between them after 2 hours.



Solutions

B16.1 The rectangle must have length 2x and width y. Therefore its area can be initially expressed as A(x,y) = 2xy ....(1)
However, since the width is bounded by the parabola with equation $y = 12 - x^2$ (i.e. the height of the rectangle is the same as the y-coordinate on the parabola where the vertex of the rectangle intersects the parabola), we can substitute in (1) to get
$A(x) = 2x(12 - x^2) = 24x - 2x^3$

B16.2 $V = lwh$, where $l = 15 - 2x$, $w = 8 - 2x$ and $h = x$ (this becomes the height when the metal is folded to make the box)

Therefore,
$V = (15 - 2x)(8 - 2x)x$
$V = 4x^3 - 46x^2 + 120x$

B16.3 Volume = x^2h
Therefore, $4000 = x^2h$

a) We will first need to solve the above expression for h so that we can substitute an expression containing x for h in the surface area formula:

$$h = \frac{4000}{x^2}$$

Surface area, S(x,h) = Area of base + Area of the four sides = $x^2 + 4xh$.

Substituting for h gives $S = x^2 + 4x\left(\frac{4000}{x^2}\right) = x^2 + \frac{16000}{x}$

To reinforce skills with rational exponents and to help prepare students for work in calculus. In calculus they will use the Power Rule to find the

Resources

Outcomes

SCO: By the end of Mathematics 3103 students will be expected to:

B16 simplify problem situations involving functions of two or more variables to functions of one variable.

Elaboration - Instructional Strategies/Suggestions

Worthwhile Tasks for Instruction and/or Assessment

(Continued from page 83)

b) Solve $4000 = x^2h$ for x:

$$x^2 = \frac{4000}{h} = \frac{5h^2}{2}$$

$$x = \sqrt{\frac{4000}{h}} = \frac{20\sqrt{10}}{\sqrt{h}} = \frac{20\sqrt{10h}}{h}$$

Substituting for x in $S(x,h) = x^2 + 4xh$ gives

$$S(h) = \left(\frac{20\sqrt{10h}}{h}\right)^2 + 4\left(\frac{20\sqrt{10h}}{h}\right)h = \frac{4000}{h} + 80\sqrt{10h}$$

Using rational exponents gives

$$S = 4000h^{-1} + 80\sqrt{10}h^{1/2}$$

B16.4 Using similar triangles,

$$\frac{h}{250} = \frac{r}{50}$$

$$r = \frac{h}{50}$$

Since the volume of a cone is $V(r,h) = \mathbf{a}\pi r^2 h$, then substituting for r gives

$$V(h) = \tfrac{1}{3}p\left(\frac{h}{50}\right)^2 h$$

$$V(h) = \frac{p\,h^3}{7500}$$

B16.5 By similar triangles, since $\frac{h}{2} = x$

The volume of the prism = area of the triangular base × height

$V(x,h) = ½(2x)(h)×5$

so $V(h) = 5xh$

$$= 5\left(\frac{h}{2}\right)h$$

$$= \frac{5h^2}{2}$$

Resources

Outcomes

SCO: By the end of Mathematics 3103 students will be expected to:

B16 simplify problem situations involving functions of two or more variables to functions of one variable

Elaboration - Instructional Strategies/Suggestions

Worthwhile Tasks for Instruction and/or Assessment

B16.6 Use the formula distance = rate × time.

For Plane A, its distance from St. John's is 300t (that is, 300km/h multiplied by the number of hours, *t* hours, that have passed since noon. For students who have done Physics or Chemistry, they may find it helpful to write
distance = 300km/hour × *t* hours = 300*t* km)
For Plane B, its distance from St. John's is 800km minus the distance it travels in *t* hours i.e 800 - 250t

Using the Pythagorean Theorem, $d^2 = (300t)^2 + (800 - 250t)^2$
$= 90000t^2 + 62500t^2 - 400000t + 640000$
$= 152500t^2 - 400000t + 640000$

Therefore, $d = \sqrt{152500t^2 - 400000t + 640000}$

At $t = 2$ hours,
$d = \sqrt{152500(2)^2 - 400000(2) + 640000}$
$d = \sqrt{450000}$
$d = 300\sqrt{5}km \approx 25.9km$

Resources