Unit V - Functions: Compositions and Inverses (10-15%)

Suggested Instruction and Assessment Time for this Unit: 4-6 classes(Assumes 55-60 minute classes)

Outcomes

SCO: By the end of Mathematics 3103 students will be expected to:

B17 find the composite rule, f(g(x)) or g(f(x)), given two functions f(x) and g(x), and perform calculations involving the composite of two functions in either algebraic or graphical form

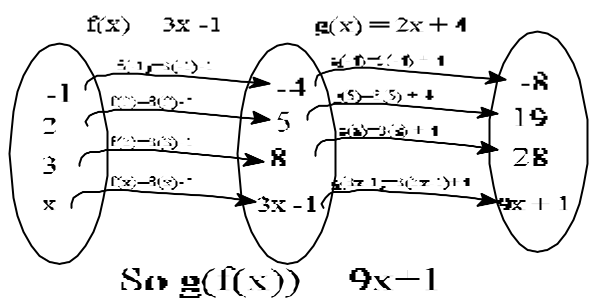
Elaboration - Instructional Strategies/Suggestions

B17 Note this outcome asks students to find the composite *rule*. They are not expected to find the domain and range of the composite rule (unless time permits and a teacher provides enrichment in this area).

This outcome is a natural extension of outcome B16, unit 4 since composition involves replacing a variable with an expression. Students are sometimes challenged in understanding why it is necessary to replace a variable with another expression containing the same variable. For example,

- $f(x) = 2x - 1$ and $g(x) = 3x + 1$,
  then $f(g(x)) = f(3x + 1) = 2(3x + 1) - 1 = 6x + 1$;
  students sometimes ask why we replace x with 3x + 1.

Students must understand that functions represent a rule e.g. $f(x) = 2x - 1$. The replacement value for x must first be doubled and then 1 must be subtracted. They may find it helpful to verbally express the operations (in the order in which they would be used!). A "bubble" or mapping diagram is helpful such as the one shown.



Each value or expression from a given "bubble" must be substituted into the next function to get the value or expression for the next bubble. Function composition can usually save a lot of unnecessary calculation when a number of values for the variable have to be used e.g. for the diagram above, to find g(f(2)) we could first find f(2) = 5 and then find g(5) = 19 OR since the composed function rule is $h(x) = g(f(x)) = 9x+1$, we could simply calculate $h(2) = 9(2) + 1 = 19$.

Familiarity with the alternate notation, $f \circ g(x)$, for composite functions should be developed. Sometimes a change of notation can be a great obstacle for students when in fact they understand the underlying concepts. It is therefore very important that they use alternative notations and translate back and forth from one notation to another.

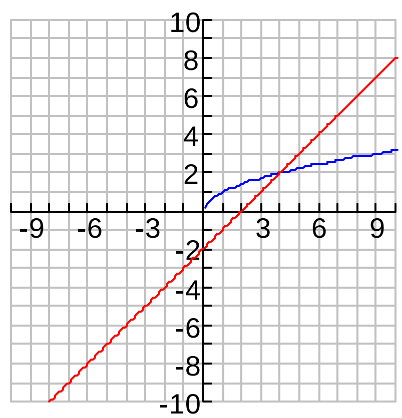
Worthwhile Tasks for Instruction and/or Assessment

Mental Math

B17.1 Given f(x) = 7x - 2 and g(x) = $6x^2$ - 1, find f(g(2)).

B17.2 The curve is f(x) and the line is g(x) in the diagram shown. Find each of the following:

a) What is g(f(9))?
b) What is f(g(2))?
c) Explain why it would be impossible to find f(g(1)) in this situation.



Pencil and Paper

B17.3 Given f(x) = $2x^2$ - 5 and g(x) = x + 4, find f(g(x)).

B17.4 Given *f(x) = 4x - 5* and , determine if *f(g(x)) = g(f(x))*.

B17.5 Given $f(x) = x + 1$, $g(x) = \sqrt{x}$, $h(x) = x^2$, and $k(x) = 3x$, match each of the expressions on the left to the appropriate composition of functions on the right:

a) $(x + 1)^2$
b) $\sqrt{3x+1}$
c) x
d) $3(x + 1)$
e) $9x^2$

i) *k(h(g(x)))*
ii) *k(k(h(x)))*
iii) *g((h(x)))*
iv) *g(f(k(x)))*
v) *f(k(x))*
vi) *h(f(x))*
vii) *f(h(x))*

Resources

Advanced Mathematics, Brown, Richard and Robbins, David. p.124-130

Mathematics 12 (National Edition), p.14 - 17

Access (Trig), HB - All exercises

Outcomes

SCO: By the end of Mathematics 3103 students will be expected to:

B18 find the inverse of a linear or quadratic function and determine whether the inverse is a function

B19 sketch the inverse graph by interchanging coordinates

Elaboration - Instructional Strategies/Suggestions

B18/19
Students need to review:
i) the exact meaning of function which is a relation in which no two ordered pairs have the same x coordinate.
ii) the Vertical Line Test (VLT)

Outcomes B18 and B19 should be done together.

The focus here is the provision of practice in the algebra of finding inverses and then applying the vertical line test to the inverse relation to see if it is also a function. However, students can be given a function to graph with the domain already restricted, along with the inverse and asked if the inverse is a function.
Students should see that an inverse is the *opposite* or *undoing* function for a given function, and therefore it must contain all the inverses of operations used in the original function. It is a good idea to start with a discussion of inverse operations

Operation	Inverse Operation
×	÷
+	-
Squaring: $(\ )^2$	$\pm\sqrt{\ }$

To introduce the idea of inverse function, some simple examples could be used to illustrate:
i) You walk upstairs from the basement to the main floor of your house. What would you have to do to get back to where you started?

ii) You are in math class. When the bell rings, you rise from your desk, walk out the classroom door, and down the hall to your locker. In the correct sequence of steps, what would you have to do to get back to where you started?

iii) A number is doubled. What would you have to do with the answer to find the original number?

iv) A number is multiplied by 4 and then 5 is subtracted. In the correct sequence of steps, what would you have to do to get back to the number with which you started?

Worthwhile Tasks for Instruction and/or Assessment

Mental Math

B18.1 What is the inverse of each function?
a) $f(x) = 3x$ b) $g(x) = x^2$ c) $h(x) = 2x + 1$

B19.1 A function f(x) contains the points (1,3) and (-1,3) on its graph. Why would it not be appropriate to refer to the inverse using $f^{-1}(x)$ notation?

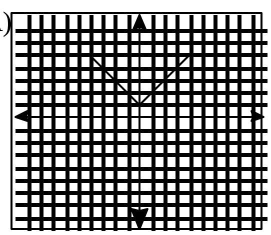
B19.2 A certain graph has y-intercept 9. What point must be on the inverse graph?

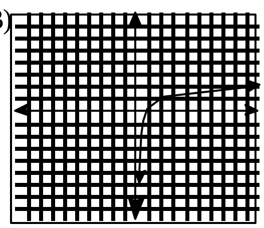
Pencil and Paper

B18.1 Find the inverse of each function below:
a) $f(x) = 5x - 6$ b) $f(x) = (x- 1)^2 - 2$
c) $y = 4x^2 - 9$

B18.2 Determine which of the inverses in B17.1 is also a function. Give a full explanation as to why you chose your answer.

B19.3 Draw the inverse of each graph below and determine if it is a function:

A)


B)


B19.4 Sketch each graph below and determine if the inverse is a function:
a) $y = 4x + 1$
b) $y = 2x - 1$
c) $y = (x + 1)(x - 2)$
d) $\frac{1}{2}(y - 3) = (x + 1)^2$ (Note: This question will involve recall of graphing using transformations on $y = x^2$ from Mathematics 1204)
e) $y = x^2 - 3, x \geq 0$
f) $y = (x + 2)(x - 3)(x-5)$

Resources

Mathematics 12 (National Edition), p.20-25

Mathematics 12: Principles & Process, p.261-262

Advanced Mathematics, Brown, Richard and Robbins, David. p.139 - 143

Access (Trig), HC - All exercises

Access (Trig), HD - All exercises

Outcomes

SCO: By the end of Mathematics 3103 students will be expected to:

B18 find the inverse of a linear or quadratic function and determine whether that inverse is a function

B19 sketch the inverse graph by interchanging coordinates

Elaboration - Instructional Strategies/Suggestions

B18 Next, discussion could center around an example involving a linear function and predicting the inverse function. For example,

Given $f(x) = 3x - 1$, the operations involved are "multiply by 3" and "subtract 1" in that order. Therefore, the inverse has the opposite order and operations i.e. "Add 1" and "divide by 3". Therefore the inverse is $y = \frac{x+1}{3}$. This method provides a quick visual check to see if a solution is correct. Students should also note that both the original and inverse relations are also functions because they pass the VLT, so the inverse in this case can also be referred to as $f^{-1}(x)$. They can also check using a few specific values for x and compose the function with its inverse to see if they get back what they started with. For example, Test $f(f^{-1}(2)) = 2 = f^{-1}(f(2))$. Students should also be able to check the more general composition

$$f(f^{-1}(x)) = x = f^{-1}(f(x))$$

Outcome B19 should be done before the process of finding an inverse algebraically. Students need to be able to visualize the graph of a given function and determine from the graph if the inverse is or is not a function.

The process of writing a function in "y =" form, interchanging x and y, then solving for y should be developed. For example,

- Given $f(x) = (x - 1)^2 + 4$,
  - a) sketch its graph and the graph of the inverse. Include at least 7 points on each graph.
  - b) determine if the inverse is a function or simply a relation.
  - c) find the inverse by interchanging x and y in the equation and then solving the equation for y.

  Solution for c)

  Write in y = form: $y = (x - 1)^2 + 4$

  Interchange x and y: $x = (y - 1)^2 + 4$

  Solve for y:

  $$(y - 1)^2 + 4 = x$$
  $$(y - 1)^2 = x - 4$$
  $$y - 1 = \pm\sqrt{x-4}$$
  $$y = 1 \pm \sqrt{x-4}$$

  Since this an inverse relation and not a function, we cannot use the f^{-1} notation.

Worthwhile Tasks for Instruction and/or Assessment

Resources

Outcomes

SCO: By the end of Mathematics 3103 students will be expected to:

B18 find the inverse of a linear or quadratic function and determine whether that inverse is a function

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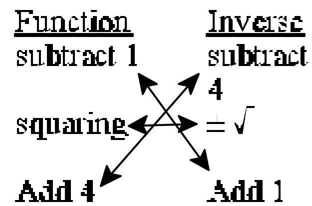
Elaboration - Instructional Strategies/Suggestions

B18/B19
Compare the order of operations for the function and its inverse:



If students have not done the Quadratic unit of Mathematics 3204/3205 prior to this study, then quadratic functions should be limited to those of the form $y = ax^2 + c$ or $y = a(x-h)^2 + k$; however, if students have done Math 3204 or if they are doing it concurrently, they can find inverses for quadratic functions of the form $y = ax^2 + bx + c$ because of their knowledge of completing the square.

Teachers may also want to use the TI-83 Plus graphing calculator and the DrawInv feature (found under the 2nd Draw menu, select 8:DrawInv. Enter the function in the Y= editor first, graph it, then access this feature). Also, to show the inverse more concretely a transparent mirror(Mira™) can be used to show that the inverse is a reflection of the function about the line y = x.

Extension

Ask students to find the inverse of an absolute value function such as $y = |2x - 3|$.

Worthwhile Tasks for Instruction and/or Assessment

Resources