Unit 5
Complex Numbers
(15 hours)

Complex Numbers

Outcomes

SCO: In this course, students will be expected to

A6 explain the connections between real and complex numbers

C27 represent complex numbers in a variety of ways

Elaboration—Instructional Strategies/Suggestions

A6/C27 Imaginary numbers were first encountered by students in their study of the roots of polynomial functions, in particular, quadratics, when exploring the nature of the roots. When examining graphs students noted that if there was no *x*-intercept on the graph the solution of the quadratic equation included the square root of a negative number. For example, the graph of $y = x^2 + 9$ does not intersect the *x*-axis which implies no real roots. However, when solving the equation $x^2 + 9 = 0$ the solution results in $x = \pm\sqrt{-3}$. Students then interpret this as a pair of roots that are not real, but imaginary.

An imaginary number is defined as a square root of a negative real number. For example, $\sqrt{-3}$ is an imaginary number. The imaginary number *i* is defined as $\sqrt{-1}$. So, $\sqrt{-3} = i\sqrt{3}$. The symbol *i* (first letter of 'imaginary') means "a number you can square to get –1 for an answer."

Thus:

$i^2 = -1$

or $i = \sqrt{-1}$

if *x* is a non-negative real number, then

$\sqrt{-x} = i\sqrt{x}$

A complex number is a number in the form *a* + *bi*, where *a* is the real part and the real number *b* is the coefficient of the imaginary part. The word "complex" mean, composed of parts.

The form *a* + *bi* is called the rectangular form of a complex number. Students may see *a* + *bi* written as an ordered pair (*a, b*), especially on calculators and computers. Later in the unit students will study complex numbers to see how they can be expressed in polar form (see p. 210). The complex numbers *a* + *bi* and *a* – *bi* are called complex **conjugates** of each other (see next two page spread).

Complex Numbers

Worthwhile Tasks for Instruction and/or Assessment

Mental Math **(C27)**

1) Write in terms of *i* in simplest form.
   a) $\sqrt{-15}$
   b) $\sqrt{-16}$
   c) $\sqrt{-48}$

Pencil and Paper **(C27)**

2) Explain why the following polynomial equations have imaginary roots:
   a) $3x^2 - 27 = 0$
   b) $3x^3 - 12x^2 + x - 4 = 0$

3) Represent these imaginary numbers in the form of $a + bi$.
   a) $\pm 3i$
   b) $i\sqrt{5}$
   c) $\sqrt{-18}$

Journal **(A6)**

4) Explain why
   a) $a + bi$, b … 0, is not a real number.
   b) the number 'three' is a complex number.
   c) $\sqrt{-17}$ is not a real number.

Suggested Resources

Complex Numbers

Outcomes

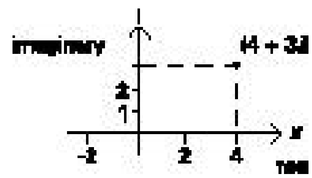
SCO: In this course, students will be expected to

C28 construct and examine graphs in the complex and polar planes

C27 represent complex numbers in a variety of ways

B9 apply operations on complex numbers both in rectangular and polar form

Elaboration—Instructional Strategies/Suggestions



C28/C27 Complex numbers can be plotted on a complex plane, called the Argand plane, or an Argand diagram. Note that the vertical axis is called the imaginary axis and the horizontal axis is the real axis. So, to plot $4 + 3i$ (sometimes noted as (4, 3), move 4 along the horizontal axis from the origin and up 3i parallel to the vertical axis. The distance a point is from the origin is called the absolute value, or modulus. Thus on the above diagram the distance from $4 + 3i$, or (4, 3) to the origin would be symbolized and calculated as follows:

if $z = 4 + 3i$, then its modulus, denoted $|z|$ can be calculated: $|z| = \sqrt{a^2 + b^2} = \sqrt{4^2 + 3^2} = \sqrt{25} = 5$

B9 Operations on complex numbers are analogous to those performed with linear binomials. For example, if $z_1 = 5 - 3i$ and $z_2 = 2 + 5i$, find

1) $z_1 + z_2$ 2) $z_1 - z_2$ 3) $z_1 \cdot z_2$

When adding and subtracting, the real parts of the complex numbers are added and subtracted as the constant terms and coefficients of variables are added and subtracted:

1) $z_1 + z_2$
$= (5 - 3i) + (2 + 5i)$
$= 7 + 2i$

2) $z_1 - z_2$
$= (5 - 3i) - (2 + 5i)$
$= 3 - 8i$

When multiplying, the distributive property is used as with the multiplication of algebraic terms.

$z_1 \bullet z_2$
$= (5 - 3i)(2 + 5i)$
$= 10 + 25i - 6i - 15i^2$
remember $i^2 = -1$
$= 10 + 19i + 15$
$= 25 + 19i$

Students should be encouraged to investigate the product of complex conjugates (($2 + 3i$) and ($2 - 3i$) are complex conjugates) to determine that their product is always a real number. For example,

$$(2 + 3i)(2 - 3i) = 4 - 9i^2$$
$$= 4 + 9$$
$$= 13$$

Students should be expected to prove this by generalizing the situation.

... continued

Complex Numbers

Worthwhile Tasks for Instruction and/or Assessment

Pencil and Paper **(C28/C27)**

1) Graph the following numbers.
   a) $3 - 2i$
   b) $-1 - 8i$
   c) $4i$
   d) the sum of 7 and $\sqrt{-25}$

Pencil and Paper **(C27/B9)**

2) Perform the indicated operations given $z_1 = 4 + 8i$ and $z_2 = -3 - i$
   a) $z_1 + z_2$
   b) $z_1 - z_2$
   c) $z_1 \cdot z_2$
   d) $\frac{z_1}{z_2}$

3) a) Find the product of $a + bi$ and its complex conjugate.
   b) Explain why the answer you just got has to be real.
   c) Use the results from above to factor the sum of 2 squares $x^2 + y^2$.
   d) Factor
      i) $x^2 + 4$
      ii) $3x^2 + 5$

4) Find a quadratic equation with real-number coefficients if one root is $2 + 3i$.

Performance **(C28/C27/B9)**

5) a) Plot $3 - 2i$ and $7 + 5i$ and call them A and B respectively.
   b) Find $A + B$ and call the sum C, then plot C.
   c) Draw vectors from the origin O, to A and then to B.
   d) Imagine the vector $\overrightarrow{OA}$ sliding along the vector $\overrightarrow{OB}$ so that O maps to B. Where will A map? Explain.
   e) If $\overrightarrow{OB}$ slides along $\overrightarrow{OA}$ so that $O \rightarrow A$, where will B map? Explain.
   f) What conjecture can you make?
   g) Test your conjecture with $(-2 + 3i) + (1 - 5i)$.
   h) Explain how using a graph would visually show subtraction. Use diagrams in your explanation.

... continued

Suggested Resources

Complex Numbers

Outcomes

SCO: In this course, students will be expected to

B9 apply operations on complex numbers both in rectangular and polar form

Elaboration—Instructional Strategies/Suggestions

... continued

B9 The product of complex conjugates is used in the dividing process:

$$\frac{3+2i}{5-3i}$$
$$=\frac{3+2i}{5-3i}\cdot\frac{5+3i}{5+3i} \quad \text{(multiply by 1 using the conjugate of the denominator.)}$$
$$=\frac{(3+2i)(5+3i)}{25+9}=\frac{15+19i-6}{34}$$
$$=\frac{9+19i}{34}=\frac{9}{34}+\frac{19}{34}i$$

By definition, $i=\sqrt{-1}$, so, $i^2=-1$. Have students examine other powers of *i* to find a pattern:

$$i^3=i^2\cdot i=-1i=-i \quad \therefore\ i^3=-i$$
$$i^4=i^2\cdot i^2=-1(-1)=1\ \therefore i^4=1$$
$$i^5=i\cdot i^4=i(1)=i\ \therefore\ i^5=i$$
$$i^6=i^2\cdot i^4=-1(+1)=-1\ \therefore i^6=-1$$
$$\text{or, } i^6=(i^2)^3=(-1)^3=-1\ \therefore i^6=-1$$

Anytime the exponent is a multiple of 4, the power is 1, so $i^{76}=1$. Remember from earlier study that a number is divisible by 4 if the number formed by the last two digits is divisible by 4. When the multiple is not 4, break the exponent down so that part of it is a multiple of 4:

$$i^{39}=i^{36}\cdot i^3=1\cdot i^3=i^3=-i\ ,$$

or break it down into a base of i^2

$$i^{39}=(i^2)^{19}\cdot i=(-1)^{19}\cdot i=-1i=-i^2$$

Complex Numbers

Worthwhile Tasks for Instruction and/or Assessment

... continued

Performance **(C27/C28/B9)**

6) Suppose $z_1 = 2+i$ and $z_2 = 1+2i$.

a) Find $z_1 z_2$.

b) Find $|z_1|, |z_2|$ and $|z_1 z_2|$.

c) Plot the points z_1, z_2 and $z_1 z_2$ on the complex plane and then connect each point to the origin (draw their position vectors). Calculate the angles made in each case between the position vectors and the horizontal axis.

d) How are the angle measures related to one another?

e) Describe in words how you would find the location of $z_3 z_4$ given $z_3 = 2 - i$, and $z_4 = -5 + 4i$.

f) Repeat all of question (6) above, but substitute the quotient $\frac{z_1}{z_2}$ for the product $z_1 z_2$.

7) Graph i, i^2, i^3, i^4 on the same complex plane. Describe what seems to be happening on the graph each time the power of i is increased by 1.

8) Suppose $z = 4 + 3i$.

a) Find iz.

b) Plot z and iz on the same complex plane. Connect each point to the origin.

c) Does $|z| = |iz|$? Explain.

d) Explain what transformation happens to z when multiplied by i.

9) What is the relationship between the modulus of a complex number and the modulus of its conjugate? Explain.

10) Express in the form $a + bi$, where a and $b \in R$.

a) $\frac{1+4i}{i}$

c) $\frac{1}{3+4i} + \frac{1}{3-4i}$

b) $\frac{i}{2+3i}$

d) $3 - 2i + \frac{1}{3-2i}$

11) Given $z = \cos 2 + i \sin 2$, and $w = \cos 2 - i \sin 2$, prove the following

a) $z + w = 2 \cos 2$

c) $z^2 = \cos 22 + i \sin 2$

b) $zw = 1$

d) $\frac{1}{1+w} = \frac{1}{2} + \frac{i}{2} \tan \frac{1}{2} q$

12) Find the number b such that $\left| \frac{2 - 3i\sqrt{5}}{6+bi} \right| = 2$.

Suggested Resources

Complex Numbers

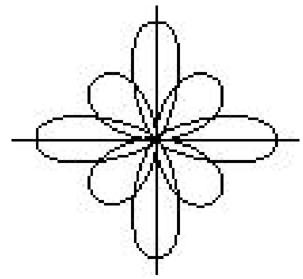
Outcomes

SCO: In this course, students will be expected to

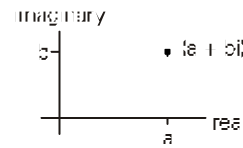
C28 construct and examine graphs in the complex and polar planes

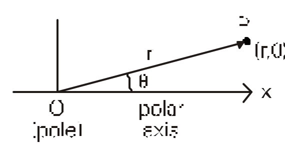
Elaboration—Instructional Strategies/Suggestions

C28 Students should now be comfortable locating points (a, b) which are the coordinates for the complex number $a + bi$, on the Argand plane. An alternate coordinate system (the polar coordinate system) is useful with multiplication and powers of complex numbers. Students will discover that this is coordinate system where surprisingly simple equations give very interesting graphs.



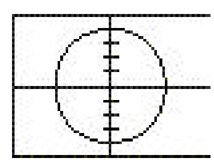
Students should have an intuitive understanding of how a point in the rectangular system can be represented by a point in the polar system and that this later would help them to understand the connections between complex numbers in rectangular form ($a + bi$) and complex numbers in polar form (see p. 211).



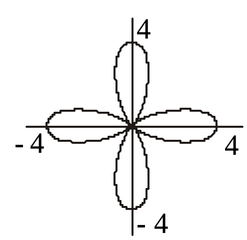


In the polar coordinate system, students should call the origin the pole O, and the fixed horizontal ray OX is called the polar axis. The position of a point P in the plane, given by the polar coordinates (r, q), has a distance $|r|$ from the origin on the vector OP. This is the image of the vector on ray OX, where $X = r$, after a rotation of q (in radians) about centre O.

- ❑ Ask students to create a table, then to graph the polar equation $r = 4$. **Solution:** This is the set of points 4 units from the origin—a circle with radius 4. There is no q in this equation, so no matter what your angle measure is, the point will be 4 units from the origin.

  Now, ask students to complete a table and plot the polar equation $r = 4 \cos 2q$. This time, as q increases, the length r varies according to the calculation $4 \cos 2q$. Students can use their graphing calculators, in polar mode (see the next two-page spread), and with PolarGC turned on in their Format menu (for tracing purposes).

... continued

Complex Numbers

Worthwhile Tasks for Instruction and/or Assessment

Performance **(C28)**

1) The coordinates $\left(\frac{p}{6},2\right)$ and $\left(-\frac{5p}{6},-2\right)$ identify the same point. Give two additional sets of polar coordinates that also name this point.
   a) Complete a table, like the one below, for the curve $r = 3 \sin 3q$ for $0° \le q \le 360°$. Graph by hand, then check using a calculator.

q	0°	5°	10°	15°	20°	25°	30°	35°	40°	45°	50°	55°
r												

60°	65°	70°	75°	80°	85°	90°	95°	100°	105°	110°	115°	...
r												

   b) If instead theta was given the range $0 \le q \le 2p$, how would the table change (describe an appropriate table)? How would you complete the settings for your calculator in order to see the same graph?

3) Using a graphing calculator:
   a) Graph the spiral $r = -0.01q$ using $0 \le q \le 2p$.
   b) Graph the equation $r = -0.01q$ using $-2p \le q \le 0$. How does this compare to the original graph? Explain why this happens.
   c) Graph the equation $r = -0.01q$ using $0 \le q \le 2p$. How does this compare to the original graph? What transformation has been performed?
   d) Graph the equation $r = 0.02q$ using $0 \le q \le 2p$. How does this compare to the original graph? What transformation has been performed?

Suggested Resources

Complex Numbers

Outcomes

SCO: In this course, students will be expected to

C28 construct and examine graphs in the complex and polar planes

Elaboration—Instructional Strategies/Suggestions

... continued

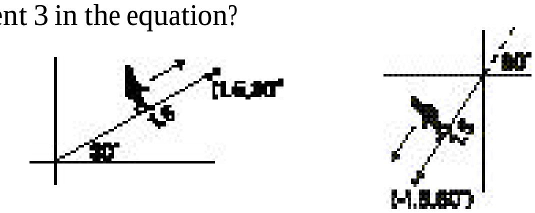
C28 If using a TI-83:

Set mode to polar, set q from 0 to 360°, set step to 5 and use *x*-values from –15 to 15, Zoom 5 to square the axes.

Example 2: Have students graph the polar equation $r = 3\cos 2q$ with $0 \le q \le 360$.

Solution: Ask students to graph this equation by hand first, then check using the calculator. Give the q -values, and ask them to complete the table (to see table on calculator, then set to PolarGC in Format menu), then plot the points. To plot a point given in polar coordinates, have students imagine themselves standing at the origin. Get them to stand at the origin, rotate through the angle (for example 30° counterclockwise from the positive ray of the horizontal axis). Then have them imagine walking straight out from the origin and placing a point at distance *r*. Students should know that this distance (*r*) is called the **modulus** of the complex number. If *r* is positive, walk forward. If *r* is negative, walk backward. As q increases from 0° degree to 360°, use this technique to locate points (q, r) on the curv**e** and then connect them with a smooth curve. What is the role of the coefficient 3 in the equation?

q	r
0°	3
10°	2.81
20°	2.3
30°	1.5
40°	0.52
50°	–52
...	...



Complex Numbers

Worthwhile Tasks for Instruction and/or Assessment

Activity **(C28)**

1) Take another good look at the equation, window, and graph for the four-petal rose graph $r = 4\cos 2q$, on p. 173. Why are there four petals? Why does the graph have the rose or flower shape? Concentrate on the connection between the trace numbers displayed on your calculator and the points. This beautiful graph comes from an equation that can be generalized as $r = a\cos nq$. In this activity, you will investigate Rose Curves, their symmetries, and the relationship between the number of petals and the value of *n*.
   a) Graph the family of curves $r = a\cos nq$ with $n = 1, 2, 3, 4, 5$, and 6. Write statements that describe the curves for even *n* and odd *n*.
   b) Graph the family of curves $r = 3\sin n\, 2$ with $n = 1, 2, 3, 4, 5$, and 6. Write statements that describe the curves for even *n* and odd *n*. How do these differ from the curves graphed in (a)?
   c) Find a way to graph a rose with only two petals. Explain why your method works.
   d) Find a connection between the polar graph $r = a\cos nq$ and the associated function graph $y = a\cos nq$. Can you look at the graph of $y = a\cos nq$ and predict the shape and number of petals in the polar graph? Explain.

Suggested Resources

Murdock, Jerald, et al, *Advanced Algebra Through Data Exploration: A graphing Calculator Approach*, Key Curriculum Press, 1998.

Complex Numbers

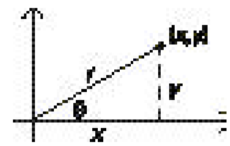
Outcomes

SCO: In this course, students will be expected to

A7 translate between polar and rectangular representations

Elaboration—Instructional Strategies/Suggestions

A7 Until now, students have used $z = a + bi$ to represent the complex number z. This is called the Cartesian or rectangular form of z. Students will now learn that z can be expressed by using its modulus $|z| = r$ and its argument θ. Ask students to apply the Pythagorean theorem to the figure below to find r and θ.



$r^2 = x^2 + y^2$ or $r = \sqrt{x^2 + y^2}$.

Also, $\tan\theta = \frac{y}{x}$ or $\theta = \tan^{-1}\frac{y}{x}$.

Since r represents a length (modulus), the distance $\sqrt{x^2 + y^2}$ will be r. The actual value of θ depends on the quadrant in which the point (x, y) is located. These two equations allow students to convert between polar and rectangular forms.

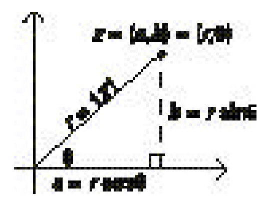
A point representing the complex number $z = a+bi$ can be given either in rectangular coordinates (a, b) or in polar coordinates (r, θ).

rectangular form:

$z = a+bi$

since $a = r\cos\theta$

and $b = r\sin\theta$



polar form:

$z = r\cos\theta + (r\sin\theta)i$

$= r(\cos\theta + i\sin\theta)$

or, $z = r\,\mathrm{cis}\,\theta$ (pronounced "r siss theta.")

For example, 3 cis 20° is the abbreviation for $3(\cos 20^\circ + i\sin 20^\circ)$.

Many calculators have keys that enable students to convert from rectangular form to polar form and vice versa. For example, using the TI-83 in degree mode:

Enter a complex number in rectangular form.

On the home screen enter $\Rightarrow 4+3i$

To convert to polar form:

Use MATH menu, cursor to CPX.

Hit 7:Polar enter $\longrightarrow 5e^{\wedge}(36.87i)$. This means 5cis36.87°.

Students are expected to do this without technology: Convert $4 + 3i$ to polar form. To get the modulus:

$r = \sqrt{a^2 + b^2} = \sqrt{4^2 + 3^2} = 5$

to get the angle measure:

$\tan\theta = \frac{3}{4} \Rightarrow \theta = \tan^{-1}\left(\frac{3}{4}\right) \doteq 36.87^\circ$

$\therefore 4 + 3i \Rightarrow 5cis36.87^\circ$

C onvert 2cis 47° to rectangular form:

$2(\cos 47^\circ + i\sin 47^\circ)$

$= (2\cos 47^\circ) + (2\sin 47^\circ)i$

$= 1.3639 + 1.4627i$

Complex Numbers

Worthwhile Tasks for Instruction and/or Assessment

Pencil and Paper **(A7)**

1) Express each complex number in polar form.

   a) $1 + i$ c) -3

   b) i d) $-\sqrt{3} - i$

2) Express each complex number in rectangular form.

   a) $5\text{cis}90°$ c) $4\text{cis}45°$

   b) $3\text{cis}p$ d) $\sqrt{6}\text{cis}130°$

3) Express $-2 + 2i$ in polar form. Sketch a graph as part of your solution.

4) Express in polar form, using the smallest non-negative value for the argument.

   a) $\frac{1}{2}\left(\sqrt{3} - i\right)$ d) $-\frac{1}{2} + \frac{\sqrt{3}}{2}i$

   b) $1 - i$ e) $-\frac{1}{2} - \frac{\sqrt{3}}{2}i$

   c) $10\sqrt{3} - 10i$ f) $-2 + 2i$

5) Change to rectangular form

   a) $6(\cos 150° + i \sin 150°)$ d) $\sqrt{18}\text{cis}\frac{5p}{3}$

   b) $2\left(\cos\frac{p}{3} + i\sin\frac{p}{3}\right)$ e) $5\text{cis}\frac{p}{3}$

   c) $6\text{cis}90°$ f) $3\sqrt{2}\text{cis}315°$

Performance **(A7)**

6) Show using a graph that $-5 - 3i$ and $5.83\ \text{cis}(-149°)$ are the same.

Journal **(A7)**

7) What does the expression $2\text{cis}\frac{p}{4}$ mean?

Suggested Resources

Complex Numbers

Outcomes

SCO: In this course, students will be expected to

B9 apply operations on complex numbers both in rectangular and polar form

B10 develop and apply DeMoivre's Theorem for powers

Elaboration—Instructional Strategies/Suggestions

B9 When students are asked to multiply complex numbers, they can either perform the operation using rectangular form as discussed on p. 202 or change to polar form and multiply in polar form.

For example, multiply (2+3*i*)(3+*i*). Students could change each factor to polar form:

$2+3i \rightarrow polar:$	$3+i \rightarrow polar:$
$r=\sqrt{2^2+3^2}=\sqrt{13}$	$r=\sqrt{3^2+1^2}=\sqrt{10}$
$\tan q=\frac{3}{2}$	$\tan q=\frac{1}{3}$
$q=56.3^o$	$q=18.4^o$
$\therefore \sqrt{13}\text{cis}56.3^o$	$\therefore \sqrt{10}\text{cis}18.4^o$

then multiply $\sqrt{13}\cos 56.3° + i\sin 56.3°$ by $\sqrt{10}\cos 18.4° + i\sin 18.4°$.

By doing an introductory activity like the one on the next page, students should learn that there is a quick way to do this (e.g., multiply the *r*-values and add the arguments).

Since $\sqrt{13}\times\sqrt{10}=\sqrt{130}$ and $\text{cis}56.3^o + cis18.4^o = cis74.7^o$

then $\sqrt{13}\text{cis}74.7^o \times \sqrt{10}\text{cis}18.4^o$

$=\sqrt{13}\times\sqrt{10}(\text{cis}(56.3^o+18.4^o))$

$=\sqrt{130}\text{cis}74.7^o$

Students should also be able to convert from polar form to rectangular form. For example,

$$\sqrt{130}\text{cis}74.7° = \sqrt{130}(\cos 74.7° + i\sin 74.7°)$$

$$= \sqrt{130}(0.263873 + i(0.9645574)) \doteq 3+11i$$

B10 In general then, when squaring a complex number in polar form,

$\left(\sqrt{a}\text{cis}q^2\right)$

$=\left(\sqrt{a}\text{cis}q\right)\left(\sqrt{a}\text{cis}q^2\right)$

$=\sqrt{a}^2\text{cis}(q+q)$

$=a\,\text{cis}2q$

This generalizes to $(rcisq)^2 = r^2cis2q$ and then to $(rcisq^n) = r^n cisnq$. This is known as **DeMoivre's Theorem** for powers.

Complex Numbers

Worthwhile Tasks for Instruction and/or Assessment

Activity **(B9)**

1) In this activity, students will discover a pattern involving multiplication of complex numbers written in polar form. This important discovery will allow students to easily multiply and divide complex numbers, and raise them to any power.
   a) Multiply each pair of complex numbers and write your answer in $a + bi$ form. (Remember $i^2 = -1$.)
      i) $(2+3i)(3+i)$
      ii) $(1+4i)(3-2i)$
      iii) $(-1+2i)(3-4i)$

   b) Convert the first pair of numbers $(2 + 3i)$ and $(3 + i)$ to polar form. Convert the product of these numbers (found in (i) above), to polar form. Do this for each product in part (a). By examining the polar form, find a relationship between the angles of the two factors and the angle of their product. This relationship should be true for all three problems. Describe the relationship between the r-values (absolute values) of the factors and the r-value of the answer.

Pencil and Paper **(B9/B10)**

2) Perform the indicated operations. You may want to change some of the numbers to polar form but express your answer in rectangular form.
   a) $(3+i\sqrt{12})^3$
   b) $3(\cos 28° + i\sin 28°)^{-1}$
   c) $(4-4i)(3+2i)^{-1}$
   d) $\sqrt{6}\text{cis}36° \cdot 2\text{cis}54°$
   e) $(2-3i)(4+5i)$
   f) $4i(3-7i)(2+5i)^2$

3) Show that $z^3 = 1$ if $z = \frac{1}{2} + i\frac{\sqrt{3}}{2}$.

4) Use DeMoivre's Theorem to evaluate: (answer in rectangular form)
   a) $(-2-i)^{12}$
   b) $(i-\sqrt{5})^{10}$
   c) $(2\text{cis}240°)^7$

Performance **(B9)**

5) If $A = \cos 2 + i \sin 2$ and $B = \cos 2 - i \sin 2$, show:
   a) $\cos q = \frac{1}{2}(A+B)$
   b) $\sin q = \frac{1}{2i}(A-B)$
   c) $AB = 1$
   d) $A^n + B^n = 2\cos(nq)$
   e) $A^n - B^n = 2\cos(nq)$
   f) $A^n - B^n = 2i\sin(nq)$

6) If z_1 and z_2 are two complex numbers, where $z_1 + z_2 = 3 + 3i$ and $z_1 \times z_2 = -2 + 6i$, find z_1 and z_2.

Suggested Resources