Mathematics 3205
[August 2006]

PART I - Total Value: 50%

Item	Item
1-A	26-D
2-D	27-B
3-B	28-B
4-B	29-D
5-B	30-B
6-C	31-A
7-D	32-B
8-B	33-B
9-A	34-A
10-C	35-C
11-A	36-B
12-C	37-A
13-D	38-C
14-A	39-A
15-D	40-D
16-C	41-D
17-A	42-C
18-C	43-A
19-D	44-B
20-D	45-D
21-A	46-C
22-B	47-A
23-D	48-A
24-A	49-B
25-D	50-B

PART II—Key…Total Value: 50%

Answer **ALL** items in the space provided. Show **ALL** workings.

Value

4 51. Algebraically determine the **EXACT** roots in simplest form for

$$\frac{5x+2}{x+3}=\frac{2x}{x+3}-\frac{x}{x-3}.$$

$$(x+3)(x-3)\cdot\left(\frac{5x+2}{x+3}=\frac{2x}{x+3}-\frac{x}{x-3}\right)$$

$$\cancel{(x+3)}(x-3)\frac{5x+2}{\cancel{x+3}}=\cancel{(x+3)}(x-3)\frac{2x}{\cancel{x+3}}-(x+3)\cancel{(x-3)}\frac{x}{\cancel{x-3}}$$

$$5x^2+2x-15x-6=2x^2-6x-x^2-3x$$

$$5x^2-13x-6=x^2-9x$$

$$4x^2-4x-6=0$$

$$2x^2-2x-3=0$$

$$x=\frac{-(-2)\pm\sqrt{(-2)^2-4(2)(-3)}}{2(2)}=\frac{2\pm\sqrt{4+24}}{4}$$

$$x=\frac{2\pm\sqrt{28}}{4}=\frac{2\pm2\sqrt{7}}{4}$$

$$\boxed{x=\frac{1\pm\sqrt{7}}{2}}$$

4 52. Two numbers have a difference of 24. Set up a quadratic function and use it to find the numbers if the result of adding the sum of the two numbers and the product of the two numbers is a minimum.

1st # $= x$

2nd # = $x-24$

$$f(x)=\left(x+(x-24)\right)+x\left(x-24\right)$$

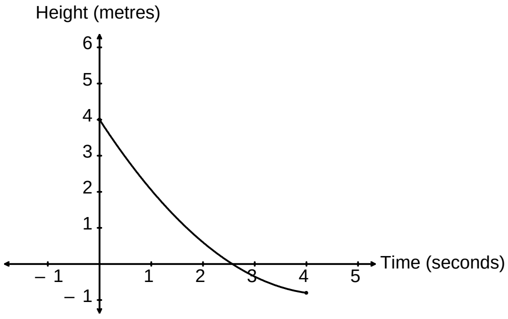
$$=2x-24+x^2-24x$$

$$=x^2-22x-24$$

$$x=\tfrac{-b}{2a}=\tfrac{-(22)}{2(1)}=11$$

$\boxed{\text{1st \# = 11, 2nd \# = } -13}$

4 53. A seagull dives from a cliff to retrieve a fish which is just below the surface of the ocean. The graph and table show the height in metres of the seagull above sea level over a 5 second period. Algebraically determine the quadratic function that defines the height of the seagull above sea level *t* seconds after leaving the cliff.



T	0	1	2	3	4	5
H	4	2.05	0.6	–0.35	–0.8	–0.75

2.05 0.6 –0.35 –0.8 –0.75

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–1.45 –0.95 0.45 –0.05

\/ \/ \/

D_2 = 0.5 0.5 0.5

$$D_2 = 2a = 0.5$$

$$\boxed{a = 0.25}$$

$$2.05 = 0.25(1)^2 + b(1) + 4$$

$$-1.95 = 0.25 + b$$

$$\boxed{-2.2 = b}$$

$$H = at^2 + bt + c$$

$$4 = 0.25(0)^2 + b(0) + c$$

$$\boxed{4 = c}$$

$$\boxed{H = 0.25t^2 - 2.2t + 4}$$

2 54(a). The height of a golf ball, *h*, in metres, *t* seconds after being struck, is given by the function $h(t) = 28t - 4.9t^2$. What is the instantaneous rate of change in the height of the golf ball at *t* = 2 seconds?

IROC

$$\frac{h(2.1)-h(2)}{2.1-2} = \frac{37.191-36.4}{0.1} = \frac{0.791}{0.1} = 79.1$$

$$\boxed{\text{The ball is gaining height at about 79.1 m/s at t = 2 seconds.}}$$

2 (b). A balloon with a diameter of 10 cm is deflating such that the radius is decreasing at a rate of 5 mm/sec. Determine the function that represents the volume of the balloon at any given instant. Note: $V_{\text{sphere}} = \frac{4}{3}\pi r^3$.

diameter = 10 cm, radius = 5 cm, 5 mm/sec = 0.5 cm/sec

$\boxed{\text{V}=\frac{4}{3}\pi(5-0.5t)^3,\ t \text{ in sec., } V \text{ in cm}^3}$ or $\boxed{\text{V}=\frac{4}{3}\pi(50-5t)^3,\ t \text{ in sec., } V \text{ in mm}^3}$

3 55. Algebraically solve: $2 \cdot 2^{2x} - 11 \cdot 2^{x} + 5 = 0$.

Let $y = 2^{x}$

$$2y^{2} - 11y + 5 = 0$$
$$(2y-1)(y-5) = 0$$

$$2y - 1 = 0 \quad \text{or} \quad y - 5 = 0$$
$$y = \tfrac{1}{2} \quad \text{or} \quad y = 5$$

$$2^{x} = \tfrac{1}{2}$$
$$\boxed{x = -1}$$

$$2^{x} = 5$$
$$\log 2^{x} = \log 5$$
$$x \log 2 = \log 5$$
$$\boxed{x = \tfrac{\log 5}{\log 2} \doteq 2.32}$$

4 56. Algebraically solve: $\log_{3}\left(2x^{2} - x\right) - \log_{3}\left(x + 2\right) = 1$.

$$\log_{3}\left(\frac{2x^{2} - x}{x + 2}\right) = 1$$
$$3^{1} = \tfrac{2x^{2}-x}{x+2}$$
$$3x + 6 = 2x^{2} - x$$
$$0 = 2x^{2} - 4x - 6$$
$$0 = x^{2} - 2x - 3$$
$$0 = (x-3)(x+1)$$

$$\boxed{x = 3} \quad \text{or} \quad \boxed{x = -1}$$

Verify:

$$\log_{3}(2 \cdot 3^{2} - 3) - \log_{3}(3 + 2) \overset{?}{=} 1$$
$$\log_{3}(15) - \log_{3}(5) \overset{?}{=} 1$$
$$\log_{3}\left(\tfrac{15}{5}\right) = \log_{3}\left(3\right) = 1$$

$$\log_{3}(2 \cdot (-1)^{2} - (-1)) - \log_{3}((-1) + 2) \overset{?}{=} 1$$
$$\log_{3}(3) - \log_{3}(1) \overset{?}{=} 1$$
$$\log_{3}(3) - 0 = 1$$

4 57. A radioactive element with an initial mass of 100 g decays to 70.7 g in 25 years. Write a function to model this situation and use it to determine the half-life of the element.

$$y = ab^{\frac{x}{h}}$$
$$y = 100\left(\tfrac{1}{2}\right)^{\frac{25}{h}}$$

$$70.7 = 100\left(\tfrac{1}{2}\right)^{\frac{25}{h}}$$
$$\frac{70.7}{100} = \frac{100(0.5)^{\frac{25}{h}}}{100}$$
$$0.707 = (0.5)^{\frac{25}{h}}$$
$$\log 0.707 = \tfrac{25}{h}\log(0.5)$$
$$h = \frac{25\log(0.5)}{\log 0.707} \doteq \boxed{\text{50 years.}}$$

Alternative Solution

$$y = ab^{\frac{x}{c}}$$
$$y = 100\left(0.707\right)^{\frac{x}{25}}$$

$$50 = 100\left(0.707\right)^{\frac{x}{25}}$$
$$\tfrac{1}{2} = \left(0.707\right)^{\frac{x}{25}}$$
$$\log 0.5 = \tfrac{x}{25}\log(0.707)$$
$$x = \frac{25\log(0.5)}{\log 0.707} \doteq \boxed{\text{50 years.}}$$

4 58. On a television show, a homicide victim was found in a warehouse where the room temperature was a constant $20°\text{C}$. Based on the table below, write a function to show the relationship between body temperature and time, and use it to find the time, to the nearest hour, at which the body temperature would have been $22°\text{C}$ had the victim remained in the warehouse.

Time (hours)	*Body Temp* (°C)
1	33.6
2	30.88
3	28.704
4	26.9632

Time (hours)	*Adj. Body Temp* (°C)
1	13.6
2	10.88
3	8.704
4	6.9632

Ratio
0.8
0.8
0.8

To find 'a', $13.6 \div 0.8 = 17$

$$y = 17(0.8)^x + 20$$

$$22 = 17(0.8)^x + 20$$
$$\tfrac{2}{17} = (0.8)^x$$
$$x = \frac{\log\left(\tfrac{2}{17}\right)}{\log(0.8)} \doteq 9.59$$

$$\boxed{\text{The body temperature was 22°C about 10 hours after death.}}$$

3 59. Algebraically show that the line $x - 2y - 4 = 0$ passes through the centre of the ellipse $25x^2 + 4y^2 - 100x + 8y + 4 = 0$.

$$25x^2 - 100x + 4y^2 + 8y = -4$$

$$25(x^2 - 4x) + 4(y^2 + 2y) = -4$$

$$25(x^2 - 4x\ \underline{+4}) + 4(y^2 + 2y\ \underline{+1}) = -4\ \underline{+100}\ \underline{+4}$$

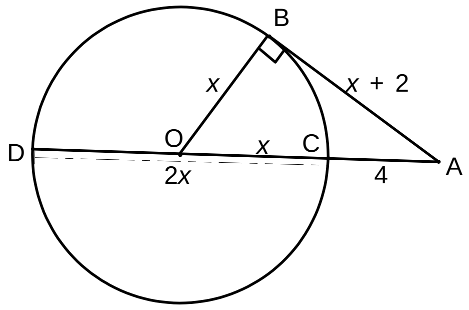
$$25(x-2)^2 + 4(y+1)^2 = 100$$

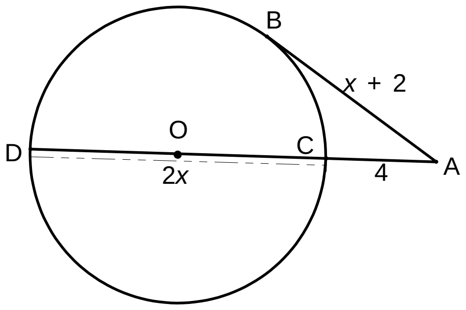
Centre: $(2, -1)$

Test in $x - 2y - 4 = 0$:

$$2 - 2(-1) - 4 \overset{?}{=} 0$$
$$2 + 2 - 4 = 0$$
$$0 = 0$$

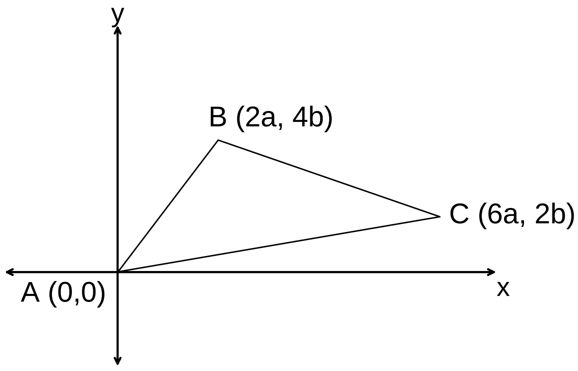
4 60. Given $\overline{AB}$ is tangent to the circle with centre O as shown, determine the value of x.





$$x^2+(x+2)^2=(x+4)^2$$
$$x^2+x^2+4x+4=x^2+8x+16$$
$$x^2-4x-12=0$$
$$(x-6)(x+2)=0$$
$$\boxed{x=6} \text{ or } \cancel{x=-2}$$

4 61. Using coordinate geometry, prove that the line segment joining the midpoints of $\overline{AB}$ and $\overline{BC}$ is equal to half the length of $\overline{AC}$.



Mdpt. of $\overline{AB}=\left(\dfrac{2a}{2},\dfrac{4b}{2}\right)=\left(a,2b\right)$

Mdpt. of $\overline{BC}=\left(\dfrac{8a}{2},\dfrac{6b}{2}\right)=\left(4a,3b\right)$

Length of segment joining midpoints:

$\sqrt{(4a-a)^2+(3b-2b)^2}=\sqrt{a^2+b^2}$

Length of segment $\overline{AC}$:

$\sqrt{(6a-0)^2+(2b-0)^2}$

$\sqrt{36a^2+4b^2}$

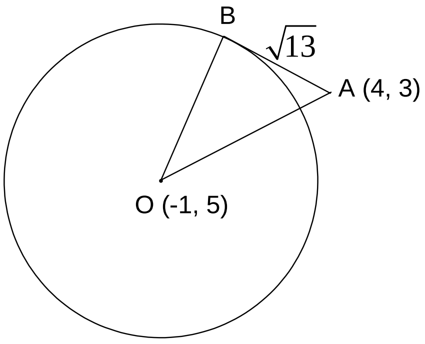
$\sqrt{4(9a^2+b^2)}$

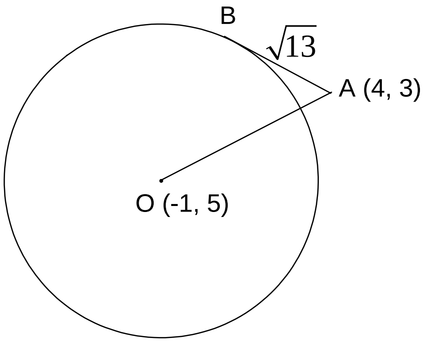
$2\sqrt{(9a^2+b^2)}$

Since, $\sqrt{9a^2+b^2}=\frac{1}{2}\left(2\sqrt{9a^2+b^2}\right)$

Therefore, the length of the segment joining the midpoints of $\overline{AB}$ and $\overline{BC}$ is half the length of $\overline{AC}$.

4 62. In the circle with centre O, $\overline{AB}$ is tangent to the circle at B. Determine the equation of the circle in general form.





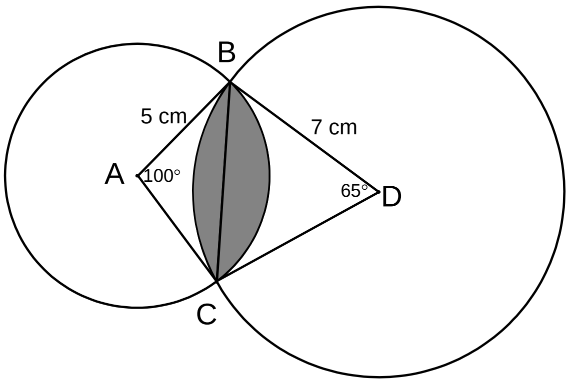
Construct $\overline{OB}$ (radius) giving AOB as a right triangle.

$$\overline{AO} = \sqrt{(4+1)^2 + (3-5)^2}$$
$$= \sqrt{25+4}$$
$$= \sqrt{29}$$

$$\left(\overline{OB}\right)^2 = \left(\sqrt{29}\right)^2 - \left(\sqrt{13}\right)^2$$
$$\left(\overline{OB}\right)^2 = 29-13$$
$$\left(\overline{OB}\right)^2 = 16$$
$$\overline{OB} = 4$$

$$(x+1)^2 + (y-5)^2 = 16$$
$$x^2 + 2x + 1 + y^2 - 10y + 25 - 16 = 0$$
$$\boxed{x^2 + y^2 + 2x - 10y + 10 = 0}$$

4 63. Given two circles with centres A and D as shown, determine the area of the shaded region to the nearest tenth of a cm^2.



Area of $\Delta ABC = \frac{1}{2}(5)(5)\sin 100° \doteq 12.31$

Area of Sector ABC $= \frac{100}{360}\pi(5)^2 \doteq 21.82$

Area of $\Delta DBC = \frac{1}{2}(7)(7)\sin 65° \doteq 22.20$

Area of Sector DBC $= \frac{65}{360}\pi(7)^2 \doteq 27.79$

$$21.82 - 12.31 \doteq 9.51$$
$$27.79 - 22.20 \doteq 5.59$$

$$\boxed{\begin{array}{l}\text{Area of Shaded Region is approx.}\\ 9.51 + 5.59 = 15.1 \text{ cm}^2\end{array}}$$